

p-Sidon Sets and a Uniform Property

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Let G be a compact abelian group with dual group Γ . Denote by $L^p(E)$ and $M(E)$ the usual spaces of Haar-measurable functions and bounded regular Borel measures, respectively, which are supported on the subset E of G or Γ . The Haar measure on G is normalized and its dual is the Haar measure on Γ . Let $\hat{\phi}$ denote the Fourier or Fourier-Stieltjes transform of the function or measure ϕ . A subset $E \subset \Gamma$ is said to be *p*-Sidon for some $1 \leq p < 2$ (not interesting for $p \geq 2$) if there is an $\alpha > 0$ such that $\|\hat{\phi}\|_p \leq \alpha \|\phi\|_\infty$ for all trigonometric polynomials ϕ on G with $\text{supp } \hat{\phi} \subset E$. This is equivalent to the dual statement: E is *p*-Sidon if and only if $L^q(E) \subset M(G)|_E$, where $1/p + 1/q = 1$ and " $|_E$ " denotes restriction to E . Hereafter p and q will *always* be as above.

The concept of a *p*-Sidon set was independently introduced in [2, 4, and 5] as a natural generalization of the classical Sidon sets (*i.e.*, 1-Sidon sets). In each of these articles, the various equivalent definitions for *p*-Sidon sets are given. They correspond to the classical equivalent definitions of a Sidon set as presented in [8, Theorem 5.7.3]. In [5], Hahn extends a theorem of J. P. Kahane to give the best known necessary conditions for a set to be *p*-Sidon when $\Gamma = \mathbf{Z}$, the integers. Edwards and Ross present the most comprehensive treatment of the subject in [4]. It is there that the first non Sidon *p*-Sidon set is constructed via an extremely ingenious application of the tensor algebraic techniques of Varapoulos. Their methods are extended in [6] to prove that the classes of all $2n/(n + 1)$ -Sidon sets are distinct for $n = 1, 2, \dots$. One will also find in [6] all known non Sidon *p*-Sidon sets to date (except for unions with finite sets). For a somewhat more skillful application of the Varapoulos techniques to this problem, we refer the reader to [1].

In this paper, we adapt an idea of Rider [7] in defining the class of uniformizable *p*-Sidon sets. The class is, by design, closed under finite unions. Of course, its members are *p*-Sidon sets. Our main result is that Sidon sets are uniformizable *p*-Sidon sets for all p . Its proof is a variant of Drury's famous technique which resembles most closely the approach found in [3]. As a corollary we prove that the union of a Sidon set with any *p*-Sidon set is again *p*-Sidon, thus enabling one to exhibit many new non Sidon *p*-Sidon sets. We conclude with a slight extension of the results in [6], using an argument similar to the one presented there, and a list of open questions.