

# On the Operator Equation $HT + TH = 2K$

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**1. Introduction.** Let  $H$  and  $K$  be bounded operators on a Hilbert space  $\mathcal{H}$ , and consider the equation

$$(*) \quad HT + TH = 2K$$

with  $T$  in  $B(\mathcal{H})$ . This equation (often written with a minus instead of plus) has a long story starting in matrices and ending up in Banach algebras (see [7], [8], [3], [2]). It was solved by E. Heinz in [4, Satz 5] for the case where the numerical range of  $H$  is bounded away from zero. Our setting is slightly different since we demand  $H \geq 0$ , but allow zero in the spectrum of  $H$  (but not as an eigenvalue). We present two formulas for the (unique) solution, one where  $T$  is a differential, the other where  $T$  is a barycenter of operators unitarily equivalent to  $H^{-1/2}KH^{-1/2}$ . The emphasis is thus on  $B(\mathcal{H})$  as a  $C^*$ -algebra, not just a Banach algebra.

The corresponding equation for functionals  $\varphi$  and  $\psi$  on von Neumann algebras, i.e.  $\varphi(T \cdot) + \varphi(\cdot T) = 2\psi$ , has been studied by S. Sakai and by A.v. Daele, who in [1] reduced it to the equation (\*), but with unbounded  $H$  and  $K$ . We shall use the techniques from [1] but only for the bounded case, which is considerably simpler and consequently allows more complete answers.

**2. Barycenter solution.** Assume that  $H$  is positive and non-singular. Since (\*) is a linear equation and the transformation  $T \rightarrow HT + TH$  preserves adjoints, there is no loss of generality in assuming that  $K = K^*$ . Note that (\*) has at most one solution. For if  $HT + TH = 0$  then since the operators  $HT$ ,  $H^{1/2}TH^{1/2}$  and  $TH$  have the same spectrum (zero apart) we conclude from  $HT = -TH$  that  $H^{1/2}TH^{1/2} = 0$ , whence  $T = 0$  since  $H$  is non-singular.

**2.1. Lemma** (v. Daele). *If  $K = H^{1/2}AH^{1/2}$  with  $A$  in  $B(\mathcal{H})$ , then the solution to (\*) is*

$$T = \int_{-\infty}^{\infty} H^{it} A H^{-it} (\cosh(\pi t))^{-1} dt.$$

*Proof.* Assume first that  $H \geq \epsilon I$ ,  $\epsilon > 0$ . In that case the function  $f : z \rightarrow H^{iz+1} A H^{-iz}$  is analytic in the upper half plane. Using the fact that  $i/2$  is the