

Diffeomorphisms with Convergent Iterates

L. S. HUSCH

Let $\pi : M \rightarrow B$ be a smooth vector bundle and define T on M by $T(v) = v/2$. Note that T is a diffeomorphism of M onto itself such that $\lim_{i \rightarrow +\infty} h^i = \pi$. In this paper we explore the converse: suppose that h is a diffeomorphism of an open manifold M onto itself such that h has *convergent iterates* [i.e., $\lim_{i \rightarrow +\infty} h^i = \alpha$ exists and defines a smooth map on M], we show $\alpha : M \rightarrow \alpha(M)$ is a vector bundle and, with additional hypotheses, we show that h is “almost” topologically equivalent to the map $z \rightarrow z/2$. The difference between h and the map $z \rightarrow z/2$ is a possible twisting of the associated sphere bundle. If M is homeomorphic to Euclidean space, then the additional hypotheses are not needed and there is no twisting; as a result (Theorem 3.2) we get that if h and g are two orientation-preserving diffeomorphisms of M such that h and g have convergent iterates, $\alpha = \lim_{i \rightarrow +\infty} h^i$ and $\beta = \lim_{i \rightarrow +\infty} g^i$ with dimension $\alpha(M) \neq m - 4$ or $m - 5$, then h and g are topologically equivalent if and only if $\alpha(M)$ is homeomorphic to $\beta(M)$. These results may be considered as a generalization of, and are strongly motivated, by the results of B. V. Kerékjártó [10], T. Homma and S. Kinoshita [7] and L. Husch [9] who considered the case where M is Euclidean space and α is a constant map. If h is not smooth, then there is considerable pathology [3]. By smooth manifold and smooth maps, we mean manifolds and maps of class C^r , $r \geq 1$.

1. Fundamental Theorem. Let M be an open connected smooth m -dimensional manifold and let h be a diffeomorphism of M onto itself with convergent iterates, $\alpha = \lim_{i \rightarrow +\infty} h^i$. If $\alpha(M) = M$, then α is the identity map and, hence, h is also the identity map; henceforth, let us assume that $\alpha(M) \neq M$. Note that α is a retraction and, by [14], $\alpha(M)$ is a smooth submanifold of M and α has constant rank, say m_0 . Thus, for each $x \in \alpha(M)$, $\alpha^{-1}(x)$ is a smooth submanifold of M of dimension $n = m - m_0$. It follows easily from the implicit function theorem the definition of α and [2] that $\alpha^{-1}(x)$ is diffeomorphic to \mathbf{R}^n , Euclidean n -space. Let $M_\alpha = M - \alpha(M)$, let M_h be the orbit space of $h \mid M_\alpha$ and let $p : M_\alpha \rightarrow M_h$ be the natural projection. The proof of the following propositions as well as the proofs of Propositions 1.3 and 1.4 are contained in the next section.

Proposition 1.1. *p is a covering map and M_h is a manifold.*