

# *The Initial Value Problem for Surface Waves Under Gravity, II. The Simplest 3-Dimensional Case*

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**1. Introduction.** Free boundary problems are notoriously difficult, and the problem of surface waves in the presence of gravity—the so-called [1] “water waves”—is particularly so [2]. The first general, unsteady solution to this problem was presented in [3], where the initial value problem is solved under the hypothesis that the fluid initially fills a horizontal slab in two dimensions.

If solutions to water wave problems in two dimensions are rare (as they are: cf. [3, 4]), in three or more dimensions they are entirely absent. The purpose of the present paper is to solve the same initial value problem solved in [3], but in three (and higher) dimensions. To our knowledge, this is the first result of any kind about the full, non-linear equations of water waves in other than two dimensions.

The specific problem we consider is that of a non-viscous, incompressible fluid with a free surface, moving under gravity. Introduce a coordinate system  $(\xi_1, \xi_2, \eta)$ , with the  $\eta$ -axis pointing up, and let  $X(t) = (x_1(t), x_2(t), y(t))$  be the coordinates at time  $t$  of the particle of fluid that was initially at  $(\xi_1, \xi_2, \eta)$ . Then, the equations of motion, in Lagrange form, are

$$(1.1) \quad (X_{tt} - \bar{g}) \frac{\partial(x_1, x_2, y)}{\partial(\xi_1, \xi_2, \eta)} + \frac{1}{\rho} \nabla p = \bar{0},$$

$$(1.2) \quad \det \frac{\partial(x_1, x_2, y)}{\partial(\xi_1, \xi_2, \eta)} = 1.$$

Here,  $\partial(x_1, x_2, y)/\partial(\xi_1, \xi_2, \eta)$  denotes the Jacobian matrix of  $(x_1, x_2, y)$  with respect to  $(\xi_1, \xi_2, \eta)$  and “det” stands for determinant.  $\nabla$  is the gradient  $(D_{\xi_1}, D_{\xi_2}, D_\eta)$ ,  $p$  is the pressure, and  $\bar{g}$  is the acceleration due to gravity (pointing down). The density,  $\rho$ , is assumed to be constant and can therefore be absorbed into the pressure. Accordingly, we assume  $\rho$  to be unity in all that follows.

In this paper we look at the simplest three-dimensional problem—that in which the domain occupied by the fluid at the initial instant  $t = 0$  is assumed