

A Remark on the Residue Theorem of Bott

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1. Introduction. In [B1] and [B2] Bott showed the relationship between the characteristic numbers of an m -dimensional compact complex manifold M and the structure of a holomorphic vector field X in a neighborhood of its zero set. Specifically, he showed that if $\varphi \in H^{2m}(BU_m; \mathbf{C})$ and N is a component of the singular set of X , then this determines a complex number $\text{Res}_\varphi(N)$ such that the Chern number of M corresponding to φ , $\varphi(TM)[M]$ satisfies

$$\varphi(TM)[M] = \sum_N \text{Res}_\varphi(N).$$

The proof of this theorem works equally well for elements of degree greater than $2m$, but an interpretation of the resulting equation, involving a global integral over M on the one hand and the sum of the local residues on the other, was lacking. In this note we give such an interpretation. It involves suspending the flow of X to obtain a family of foliations on $M \times T^2$. We then show that the derivatives of the secondary characteristic classes of these foliations are essentially the sums of the real and imaginary parts of the local residues of X .

2. Review of connections and characteristic classes. We review here very briefly the relationship between connections and characteristic classes. For a more detailed account see [KN] chapter 12 and [B3].

To begin, let E be a complex vector bundle of complex dimension q over a manifold M and let T^* be the complexified cotangent bundle of M . A *connection* ∇ on E is a map from sections of E to sections of $T^* \otimes E$

$$\nabla : C^\infty(E) \rightarrow C^\infty(T^* \otimes E)$$

such that

$$\nabla(f\sigma) = df \otimes \sigma + f \nabla \sigma$$

for smooth functions f on M . The curvature K of ∇ is obtained by extending ∇ to an antiderivation on the whole complex of forms on M with values in E and setting

$$K \cdot \sigma = \nabla(\nabla \sigma).$$

Thus $K \in C^\infty(\Lambda^2 T^* \otimes \text{End } E)$ where Λ^2 is the second exterior power. If