Self Linking Numbers of Space Polygons

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If a smooth closed space curve with nowhere vanishing curvature is pushed a small distance away from itself along its principal normal vectors to a new curve X_ϵ then the linking number of X with X_ϵ is independent of ϵ for ϵ sufficiently small and this number is called the *self-linking* number of X. This notion was first studied by Călugăreanu [3] using classical analytic techniques, and subsequently by Pohl [5] who used methods involving differential forms to establish a large number of properties of this self-linking number.

The purpose of this paper is to develop the theory of self-linking for polygons in 3-space in all of its ramifications. The proofs primarily rest upon a basic projection theorem which makes it possible to compute self-linking numbers simply and effectively by projecting to a plane and totalling numbers of apparent crossings and (signed) pairs of apparent inflection points.

The first section proves the basic theorem by deformation methods similar to those of [1]. Section 2 treats linking numbers of pairs of polygons in the same spirit and obtains a finite form of the *Gauss integral.* In section 3, we obtain integral formulas for the self-linking number of a polygon including a polygonal analogue of theorems involving total torsion of closed curves. Section 4 presents the connection with the definitions of Călugăreanu and Pohl in terms of normal variations and intersections with developable surfaces associated with the curve. Section 5 treats the behavior of self-linking under deformations, and the final section indicates how these concepts can be generalized to higher dimensional polyhedra.

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1. Definition of the self-linking number for a space polygon. Let $X = (X_0, X_1, \cdots, X_{m-1})$ be a space polygon determined by a cycle of vertices in general position. We set $T_i = (X_{i+1} - X_i)/|X_{i+1} - X_i||$, the unit *tangent* vector in the direction of the ith edge (where all subscripts are to be reduced modulo m). Set $B_i = (T_{i-1} \times T_i)/||T_{i-1} \times T_i||$, the unit *binormal* vector to the oriented plane containing both edges of X at X_i.

The vectors B_i and B_{i+1} are orthogonal to T_i so $B_i \times B_{i+1}$ is a multiple of T_i. We set $f_i(X) = \pm 1$ depending on the sign of $B_i \times B_{i+1} \cdot T_i$, indicating whether

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