

The Splitting of $\mathfrak{A}(T_1 \oplus T_2)$ and Related Questions

JOHN B. CONWAY & PEI YUAN WU

If T is a bounded linear operator on a Banach space \mathfrak{X} , let $\mathfrak{A}(T)$ denote the weakly closed subalgebra of $\mathfrak{B}(\mathfrak{X})$ generated by T and the identity 1. If T_1 and T_2 are two such operators then one can ask the question, "When does $\mathfrak{A}(T_1 \oplus T_2)$ split?" That is, when does $\mathfrak{A}(T_1 \oplus T_2) = \mathfrak{A}(T_1) \oplus \mathfrak{A}(T_2)$? This paper discusses this question as well as similar splitting questions for other functors on bounded operators.

For an operator T in $\mathfrak{B}(\mathfrak{X})$ define the following objects (in addition to $\mathfrak{A}(T)$). Let $\mathfrak{A}_u(T) =$ the *uniformly closed algebra* generated by T and 1. So $\mathfrak{A}_u(T)$ is the uniform closure of $\{p(T) : p \text{ is a polynomial}\}$. Note that $\mathfrak{A}(T)$ is the weak closure of the same set. Let $\{T\}'$ denote the *commutant* of T . That is,

$$\{T\}' = \{S \in \mathfrak{B}(\mathfrak{X}) : ST = TS\}.$$

Let $\{T\}''$ denote the *double commutant* of T , $\{\{T\}'\}'$. Then $\mathfrak{A}(T) \subseteq \{T\}'' \subseteq \{T\}'$.

Denote by $\text{Lat } T$ the *lattice of invariant subspaces* of T ; that is

$$\text{Lat } T = \{\mathfrak{M} : \mathfrak{M} \text{ is a closed subspace of } \mathfrak{X} \text{ and } T\mathfrak{M} \subseteq \mathfrak{M}\}.$$

The *lattice of hyperinvariant subspaces*, denoted by $h - \text{Lat } T$, consists of all the closed subspaces \mathfrak{M} of \mathfrak{X} such that $S\mathfrak{M} \subseteq \mathfrak{M}$ whenever $S \in \{T\}'$; clearly $h - \text{Lat } T \subseteq \text{Lat } T$. Finally, put

$$\text{Alg Lat } T = \{S \in \mathfrak{B}(\mathfrak{X}) : S\mathfrak{M} \subseteq \mathfrak{M} \text{ for every } \mathfrak{M} \text{ in Lat } T\}.$$

Clearly $\mathfrak{A}(T) \subseteq \text{Alg Lat } T$; if $\mathfrak{A}(T) = \text{Alg Lat } T$ then T is a *reflexive operator* [12].

In this paper the splitting of each of the above defined functors will be investigated.

There exists in the literature a number of results of this type. Brickman and Fillmore ([1], Lemma 1) showed that if T_1 and T_2 are operators on finite dimensional spaces then $\text{Lat } (T_1 \oplus T_2) = \text{Lat } T_1 \oplus \text{Lat } T_2$ if and only if their minimal polynomials are relatively prime. We generalize this in Theorem 3.1 and Proposition 3.10 to operators of class C_0 on a Hilbert space and algebraic operators. Crimmins and Rosenthal [4] showed that if T_1 and T_2 are operators on Banach spaces and $\hat{\sigma}(T_1)$ and $\hat{\sigma}(T_2)$ (the polynomially convex hulls of their spectra)