

Jordan Representation for a Class of Nilpotent Operators

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1. Introduction. Let T denote a nilpotent linear operator on a Hilbert space of dimension n ; i.e. $T^k = 0$ for some integer k . The Jordan form for T says that T is similar to an operator W ($W = XTX^{-1}$ for some nonsingular X) for which there is an orthonormal basis $\{e_k\}$ such that $We_k = \alpha_k e_{k+1}$, $k = 1, \dots, n$, and α_k is either 0 or 1. An operator W of this form will be called a 0, 1 weighted shift. W can be written as a direct sum, $W = \sum_0^m \oplus B_k$, where $B_0 = 0$ and each 'block' B_k , $k \geq 1$, has the matrix representation

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & & 0 \\ 0 & 1 & 0 & \\ \vdots & 0 & 1 & \ddots \\ \vdots & \vdots & & \ddots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

If B_k is a $j \times j$ matrix, B_k is said to have length j . If we define $\beta_i(T)$ to be the number of blocks of length j , we then see that $\{\beta_i(T)\}$, $j = 1, \dots, n$, is a complete set of similarity invariants for T ; i.e. two nilpotent operators S and T are similar if and only if $\beta_i(S) = \beta_i(T)$ for all j .

Suppose now that \mathcal{H} is a separable infinite dimensional Hilbert space. We may define a 0, 1 weighted shift exactly as above, but there is no hope that every nilpotent bounded operator is similar to such a shift. Every 0, 1 weighted shift W satisfies the property that the range of W^k , $k \geq 1$, is closed; clearly, not every nilpotent possesses this attribute. The purpose of this paper is to show that this necessary condition is also sufficient: if T is nilpotent and the range of T^k , $k \geq 1$, is closed, then T is similar to a 0, 1 weighted shift.

The key idea behind the proof is that we can define the quantity $\beta_i(T)$ for an arbitrary nilpotent T . If W is a 0, 1 weighted shift, then it is easy to see that

$$\beta_i(W) = \dim [N_W \cap R(W^{k-1}) - N_W \cap R(W^k)]$$