

Semi-Local Taut Rings

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Introduction. A ring R is taut in case, for each prime ideal P in R , height $P + \text{depth } P \neq \{1, \text{altitude } R\}$. (Thus the concept of a semi-local taut ring is a generalization of the concept of a catenary local domain.) A number of properties of catenary local rings were proved in [5]. In this paper, it is shown that some of these properties continue to hold in the class of semi-local taut rings. Two of these results which concern a prime ideal P in a semi-local taut ring R are : R/P is taut; and, if P is non-maximal, then R_P satisfies the second chain condition.

Besides these two theorems, it is also shown that if S is an integral extension ring of a semi-local ring R such that minimal prime ideals in S contract to minimal prime ideals in R , then S is taut if and only if R is, and then, if Q is a prime ideal in S that height $Q < \text{height } Q \cap R$, then Q is a height one maximal ideal.

A subclass of taut rings of special interest is the set of taut-level rings. (R is taut-level if height $P + \text{depth } P = \text{altitude } R$ for all primes P of R .) It was previously known that a local taut-level domain satisfies the first chain condition. Herein it is shown that this can be strengthened to saying that a local taut-level ring satisfies the first chain condition; however, a semi-local taut-level ring need not satisfy the first chain condition, as an example shows. The situation concerning semi-local taut-level domains is unknown. That is, it is an open question if a semi-local taut-level domain must satisfy the first chain condition. (This implies the catenary chain conjecture: the integral closure of a catenary local domain satisfies the chain condition.)

Since both in the general study of commutative rings as well as in the particular study of various open questions such as the catenary chain conjecture, one often deals with integral extension rings of Noetherian rings, some of the above results are extended to integral extension rings of semi-local taut rings.

Definitions and notations. In this paper, all rings are commutative with 1. Semi-local implies Noetherian, and the symbol " \subset " will mean proper inclusion. We have need of the following definitions:

- (i) The ring R is *taut* if for all primes P of R either height $P + \text{depth } P = \text{altitude } R$ or height $P + \text{depth } P = 1$.