

Proper Holomorphic Mappings in \mathbf{C}^n

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1. Introduction. A proper holomorphic mapping $\phi : D_1 \rightarrow D_2$ of two bounded domains in \mathbf{C}^n is a holomorphic mapping such that sequences in D_1 converging to ∂D_1 are mapped by ϕ to sequences converging to ∂D_2 . It follows that ϕ is a branched cover of some order $\lambda \geq 1$; see [7] for the standard facts on proper maps. The general problem that I shall consider is this: When is a proper holomorphic mapping of given domains in \mathbf{C}^n necessarily a biholomorphism? For one complex variable, the map $z \mapsto z^N$ ($N > 1$) of the unit disc to itself shows that branching can indeed occur. This example lends itself to taking products: $(z_1, z_2, \dots, z_n) \mapsto (z_1^N, z_2, \dots, z_n)$ is a proper self-mapping of the unit polydisc in \mathbf{C}^n with branching order N . The lack of an obvious way to similarly write down such a proper holomorphic self-mapping of the unit ball in \mathbf{C}^n ($n > 1$) other than an automorphism has led to the conjecture that there are none. A more general conjecture is that a holomorphic proper self-mapping of a strictly pseudo-convex domain in \mathbf{C}^n ($n > 1$) is an automorphism. One might try to push this even further by considering a proper map between different strictly pseudo-convex domains. In this setting, however, the map need not be a biholomorphism. Y. -T. Siu has pointed out a simple example in \mathbf{C}^2 for non-simply connected domains; namely, let

$$D_1 = \left\{ (z, w) \in \mathbf{C}^2 : |z|^4 + \frac{1}{|z|^4} + |w|^2 < 3 \right\}$$

and

$$D_2 = \left\{ (z, w) \in \mathbf{C}^2 : |z|^2 + \frac{1}{|z|^2} + |w|^2 < 3 \right\}$$

and put $\phi : D_1 \rightarrow D_2$, $\phi(z, w) = (z^2, w)$, to get a 2-to-1 map of strictly pseudo-convex domains. Observe that ϕ is a covering projection (without branching!) in this example. This is what is likely to be true in general. In fact, if one could show that ϕ extended to be smooth on all of ∂D_1 , then a local computation of J.-E. Fornaess ([4], Chapter 1, Section 7) would apply to show that the derivative ϕ' is non-singular on ∂D_1 (hence on D_1) and thus ϕ would be a covering projection; in particular, for D_2 simply connected, ϕ would be a biholomorphism.