

On Arveson's Characterization of Hyperreducible Triangular Algebras

ERIC NORDGREN, HEYDAR RADJAVI &
PETER ROSENTHAL

W. B. Arveson's deep study of non-self-adjoint operator algebras [2] contains a number of interesting results related to the question of which algebras containing maximal abelian von Neumann algebras are reflexive. In this note we show that one of these results, the theorem on "pre-reflexivity", can be proven in a "coordinate-free" manner.

Arveson's pre-reflexivity theorem [2] states that if \mathfrak{A} is an ultraweakly closed algebra of operators on a separable Hilbert space, and if \mathfrak{A} includes a maximal abelian self-adjoint algebra, then $B \in \mathfrak{A}^*$ whenever B is reduced by every invariant subspace of \mathfrak{A} . Our statement of the theorem below will not be as strong as that of [2], in that [2] merely requires \mathfrak{A} to be ultraweakly closed; on the other hand, our proof is valid for algebras on any space, whereas [2] requires separability of the Hilbert space.

One immediate corollary of Arveson's pre-reflexivity theorem is the result of [4] that a weakly closed reductive algebra which contains a maximal abelian self-adjoint algebra (m.a.s.a. for short) is self-adjoint. Another corollary pointed out in [2] is a characterization of hyperreducible triangular algebras. Recall that Kadison and Singer [3] defined a *triangular operator algebra* to be an algebra \mathfrak{A} of (bounded, linear) operators on a Hilbert space such that $\mathfrak{A} \cap \mathfrak{A}^*$ is a m.a.s.a. The triangular algebra \mathfrak{A} is called hyperreducible if the set of projections onto its invariant subspaces generates $\mathfrak{A} \cap \mathfrak{A}^*$ (as a weakly closed algebra).

Our proof of the theorem is in the spirit of previous work on related questions, especially [1] and [4]. Our notation will conform to that in [5]. If \mathcal{H} is a Hilbert space, then $\mathcal{H}^{(n)}$ will denote the direct sum of n copies of \mathcal{H} . If A is an operator on \mathcal{H} , then $A^{(n)}$ is the operator on $\mathcal{H}^{(n)}$ consisting of the direct sum of n copies of A . For any subset \mathfrak{A} of the algebra $\mathfrak{B}(\mathcal{H})$ of all operators on \mathcal{H} , $\mathfrak{A}^{(n)} = \{A^{(n)} : A \in \mathfrak{A}\}$. The lattice of all invariant subspaces of a set \mathfrak{A} of operators will be denoted $\text{Lat } \mathfrak{A}$, and for $\text{Lat } \{A\}$ we will simply write $\text{Lat } A$.

Theorem (cf. [2]). *Let \mathfrak{A} be a weakly closed algebra of operators that includes a m.a.s.a. If every invariant subspace of \mathfrak{A} reduces an operator B , then $B \in \mathfrak{A} \cap \mathfrak{A}^*$.*