

# On A Superlinear Differential Equation

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**1. Introduction.** The equation to be considered in this paper is

$$(1.1) \quad u'' + e^{-x}f(u) = 0$$

where the function  $f$  is of class  $C^1[0, \infty)$  and is assumed to satisfy the condition:  $f(t) = tg(t)$  where  $g$  is an increasing function such that  $g(t) \rightarrow 0$  as  $t \rightarrow 0$  and  $g(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . This is often referred to as the *superlinearity* condition and is assumed in everything that follows without further mention. Since our concern is solely with positive parts of solutions there is no need to assume anything about the function  $f$  for  $t < 0$ .

Our aim in this paper is to contribute some new methods and results to the general theory concerning the relationship between the *rate of growth* of the function  $f$  and the global structure of the family of solutions of (1.1). We are concerned, in particular, with the case where the function  $f$  is very rapidly increasing. Much of the work can be extended to cover more general equations of the form

$$(1.2) \quad y'' + yF(y^2, x) = 0,$$

but for the present we restrict attention to (1.1).

For the sake of conciseness in stating our results we introduce the following terminology concerning solutions for which  $u(\alpha) = 0$  and  $u'(\alpha) > 0$ . Such a solution will be called of type *A* if there exists  $a > \alpha$  such that  $u'(a) = 0$  and in the sequel  $a$  will always be the *first* such number. A solution of type *B* will be one for which  $u'(x) > 0$  for all  $x > \alpha$  and which satisfies  $u(x) \rightarrow m < \infty$  as  $x \rightarrow \infty$ . Finally, a solution of type *C* remains increasing in  $x > \alpha$  but  $u(x) \rightarrow \infty$  as  $x \rightarrow \infty$ .

We stress that the types *A*, *B* and *C* refer only to solutions for which  $u(\alpha) = 0$ ,  $u'(\alpha) > 0$  and that all solutions of (1.1) referred to in the following theorems satisfy these conditions.

The principal results in this paper are the following.

**Theorem 1.1.** *Let  $u$  be a solution of type C and suppose that the first variation*

$$(1.3) \quad \phi'' + e^{-x}f'(u(x))\phi = 0$$

*is oscillatory on  $(\alpha, \infty)$ . Then  $u'(\alpha)$  is not an end point of any interval  $I$  with the*