

The Direct Sum of Normal Operators

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Let A and B be normal operators on a Hilbert space. It is well known that the von Neumann algebra generated by $A \oplus B$ splits into the direct sum of the von Neumann algebras generated by A and B if and only if A and B have mutually singular spectral measures. This is also equivalent to each of the statements: (i) $\{A \oplus B\}'$, the commutant of $A \oplus B$, is $\{A\}' \oplus \{B\}'$; and (ii) every reducing subspace of $A \oplus B$ is the direct sum of a reducing subspace of A and a reducing subspace of B .

In this paper we study certain "splitting" properties of $A \oplus B$ for functors which do not involve the adjoint of $A \oplus B$. For example, if $\alpha(T)$ denotes the weakly closed algebra generated by an operator T and $\text{Lat } T$ denotes its lattice of invariant subspaces then: when is $\alpha(A \oplus B) = \alpha(A) \oplus \alpha(B)$? When is $\text{Lat } (A \oplus B) = \text{Lat } A \oplus \text{Lat } B$? Actually, for normal operators these questions are equivalent because normal operators are reflexive.

The condition that the spectral measures of A and B be mutually singular is necessary in order that $\alpha(A \oplus B) = \alpha(A) \oplus \alpha(B)$, but it is not sufficient as the following example illustrates. Let $\Gamma_+(\Gamma_-)$ be the upper (lower) semi-circle of radius one about the origin in \mathbf{C} , let m be Lebesgue measure on the circle Γ and let $A(B)$ be multiplication by z on $L^2(\Gamma_+, m)$ ($L^2(\Gamma_-, m)$). Then $\alpha(A) \oplus \alpha(B)$ consists of all multiplication operators defined by elements of $L^\infty(\Gamma, m)$ while $\alpha(A \oplus B)$ only contains the multiplication operators defined by functions in H^∞ .

In this paper two sets of necessary and sufficient conditions that $\alpha(A \oplus B) = \alpha(A) \oplus \alpha(B)$ are given. One (Theorem 4.4) is operator theoretical and the other (Theorem 2.1) is phrased in terms of Sarason's characterization of the weak star closure of the polynomials in $L^\infty(\mu)$ for a measure μ on \mathbf{C} with compact support [12].

As an application, Theorems 3.1 and 3.6 give necessary and sufficient conditions that $\{V_1 \oplus V_2\}'$ and $\alpha(V_1 \oplus V_2)$ split for isometries V_1 and V_2 . We also give a characterization of a decomposition of normal operators obtained by Olin [9].

It should be pointed out that the literature contains a few examples of theorems of the type treated in this paper. Crimmins and Rosenthal [4] have shown that if A and B are arbitrary operators on a Banach space and $\sigma(A)$