

On Fractional Integrals and Derivatives in L^p

RHONDA J. HUGHES

The purpose of this note is to show that the Riemann–Liouville fractional integral acting in $L^p(0, \infty)$, $1 < p < \infty$, satisfies a certain inequality which yields information about the domains of fractional integrals and derivatives as unbounded operators in $L^p(0, \infty)$.

In [4], Hardy, Landau and Littlewood proved that for certain real-valued functions f in $L^p(0, \infty)$, $1 < p < \infty$, the Weyl fractional integral

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t) dt, \quad \alpha > 0,$$

satisfies the inequality

$$(1.1) \quad \|J^\beta f\| \leq K \|J^\alpha f\|^{(\gamma-\beta)/(\gamma-\alpha)} \|J^\gamma f\|^{(\beta-\alpha)/(\gamma-\alpha)}, \quad -L' < \alpha < \beta < \gamma < L,$$

where $K = K(p, L, L')$ and $\|\cdot\|$ denotes the norm in $L^p(0, \infty)$ (if n is a positive integer, $J^{\alpha-n} f(x) = (-1)^n (d/dx)^n J^\alpha f(x)$, and $J^0 f(x) = f(x)$); those functions f for which (1.1) is proved belong to the set $D(J^\alpha) = \{f \in L^p(0, \infty) : J^\alpha f \in L^p(0, \infty)\}$ for each α real.

Now for $f \in L^p(0, \infty)$, $\alpha \in \mathbf{C}$ (the complex numbers) with $\operatorname{Re} \alpha > 0$, let

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt,$$

the Riemann–Liouville fractional integral acting in $L^p(0, \infty)$ with maximal domain $D(I^\alpha) = \{f \in L^p(0, \infty) : I^\alpha f \in L^p(0, \infty)\}$. In Theorem 3.3 of [5] it was shown that as unbounded operators in $L^p(0, \infty)$,

$$(1.2) \quad I^\alpha I^\beta = I^{\alpha+\beta}, \quad \alpha, \beta \in \mathbf{C}, \quad \operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0.$$

The crucial result is the following:

If $f \in L^p(0, \infty)$, and $I^\beta f \in L^p(0, \infty)$, where $\operatorname{Re} \beta > 0$, then $I^\alpha f \in L^p(0, \infty)$ whenever $0 < \operatorname{Re} \alpha < \operatorname{Re} \beta$.

For α, β real this was proved by Berens and Westphal [1, Proposition 6] in the context of derivatives of fractional order in $L^p(0, \infty)$ (defined as inverses of the 1-1 operators I^α); the general result follows from some results of Fisher [3] about “boundary values” of the Riemann–Liouville fractional integral in