

# Descent and Closed Range

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**1. Introduction.** Suppose  $T : X \rightarrow Y$  is a closed, possibly unbounded, linear operator with domain  $D(T)$  in  $X$  and range  $R(T)$  in  $Y$ , where  $X$  and  $Y$  are Banach spaces. An important theorem of Kato [5, Lemma 332] states that if the defect of  $T$  is finite, then the range of  $T$  is closed. This result is generalized in Goldberg [2, IV.1.12] as follows: if  $M$  is a closed subspace of  $Y$  such that  $R(T) \oplus M$  is closed, then  $R(T)$  is closed. However, if the finite defect condition in Kato's theorem is replaced by a finite descent condition, it does not follow in general that  $R(T)$  is closed. For example, see Gustafson [3, Example 5.7], or the following Example 1. Some sufficient conditions including finite descent will be given in this paper to obtain closed range.

Throughout the following it is assumed that  $T: X \rightarrow X$  is a closed linear operator with domain and range in the Banach space  $X$ . A dense domain will be assumed only where so stated. It is also assumed that the resolvent set is non-empty. Then  $T$  has the following properties:

(i) If  $P$  is any polynomial, then  $P(T)$  is a closed linear operator [1, Theorem VII.9.7].

(ii) The null space of  $P(T)$  is closed.

(iii)  $X = D(T^i) + R(T^k)$ ,  $i, k = 0, 1, 2, \dots$  [7, 1.1]. Recall that  $\text{def}(T) = \dim X/R(T)$ ,  $\text{des}(T) = \min\{m \geq 0 : R(T^m) = R(T^{m+1})\}$ ,  $\text{nul}(T) = \dim N(T)$ , and  $\text{asc}(T) = \min\{m \geq 0 : N(T^m) = N(T^{m+1})\}$ . For other notations used, see references [2, 3, 6].

Taylor and Lay already have some results concerning finite descent and closed range. Suppose  $\lambda_0$  is a point of the spectrum of  $T$  such that  $\text{asc}(\lambda_0 - T) = p < \infty$  and  $\text{des}(\lambda_0 - T) < \infty$ . Then  $\lambda_0$  is an isolated point of  $\sigma(T)$  and is a pole of the resolvent operator  $R_\lambda(T)$  of order  $p$  [7, Theorem 2.1]. For an isolated  $\lambda_0$  in  $\sigma(T)$ ,  $\lambda_0$  is a pole of order  $p$  if and only if  $\text{asc}(\lambda_0 - T) = p = \text{des}(\lambda_0 - T)$  and  $R(\lambda_0 - T)^p$  is closed [11, Theorem 9.1].

**Example 1.** Define the nilpotent operator  $T : l_2 \rightarrow l_2$  by

$$Te_{2n} = \frac{1}{n} e_{2n-1} \quad \text{and} \quad Te_{2n-1} = 0.$$

$T$  is closed and bounded,  $R(T)$  is not closed,  $T^2 = 0$ , and  $\text{des}(T) = 2 = \text{asc}(T)$ .

Thus the conclusion of the Taylor-Lay theorem above cannot be improved from  $R(T^2)$  closed to  $R(T)$  closed.