

Spectra of Compact Perturbations of Operators

C. APOSTOL, C. PEARCY & N. SALINAS

Let \mathfrak{H} be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathfrak{H})$ denote the algebra of all bounded linear operators on \mathfrak{H} . The (norm-closed, two-sided) ideal of compact operators in $\mathcal{L}(\mathfrak{H})$ will be denoted by $\mathcal{K}(\mathfrak{H})$, and the canonical quotient map of $\mathcal{L}(\mathfrak{H})$ onto the Calkin algebra $\mathcal{L}(\mathfrak{H})/\mathcal{K}(\mathfrak{H})$ by π . The spectrum of an operator T in $\mathcal{L}(\mathfrak{H})$ will be denoted by $\sigma(T)$, and the [left, right] essential spectrum of T (*i.e.*, the [left, right] spectrum of $\pi(T)$) by $[\sigma_{l_e}(T), \sigma_{r_e}(T)]\sigma_e(T)$. Recall that the Weyl spectrum of T , denoted by $\sigma_w(T)$, is the union of $\sigma_e(T)$ and all holes Γ in $\sigma_e(T)$ whose associated Fredholm index $i(\Gamma)$ is nonzero.

The purpose of this note is to answer completely the following question: if T is an arbitrary operator in $\mathcal{L}(\mathfrak{H})$, exactly which (nonempty, compact) subsets Σ of the complex plane \mathbf{C} have the property that there exists a compact operator $K = K(\Sigma)$ in $\mathcal{K}(\mathfrak{H})$ such that $\sigma(T + K) = \Sigma$? The following facts bearing on this problem are well-known.

Proposition 1. (*cf.* [7, Prop. 1.27]). *If $T \in \mathcal{L}(\mathfrak{H})$, $K \in \mathcal{K}(\mathfrak{H})$, and $\sigma(T + K)$ is denoted by Σ , then*

- a) $\Sigma \supset \sigma_w(T)$,
- b) *the intersection of Σ and the unbounded component of $\mathbf{C} \setminus \sigma_e(T)$ is a countable (perhaps void) set of isolated points, and*
- c) *if Γ is a hole in $\sigma_e(T)$ such that $i(\Gamma) = 0$, then either $\Sigma \supset \Gamma$ or $\Sigma \cap \Gamma$ is a countable (perhaps void) set of isolated points.*

Proposition 2. (*Stampfli* [8]). *If $T \in \mathcal{L}(\mathfrak{H})$, then there exists a compact operator K in $\mathcal{K}(\mathfrak{H})$ such that $\sigma(T + K) = \sigma_w(T)$.*

Proposition 1 clearly delineates what must be settled to solve the problem posed above: if $T \in \mathcal{L}(\mathfrak{H})$, then which compact sets Σ satisfying a), b), and c) of Proposition 1 have the property that there exists a K in $\mathcal{K}(\mathfrak{H})$ such that $\sigma(T + K) = \Sigma$? The answer is perhaps mildly surprising.

Theorem 1. *Let $T \in \mathcal{L}(\mathfrak{H})$ and let Σ be any compact subset of \mathbf{C} satisfying a), b), and c) of Proposition 1. Then there exists a compact operator $K = K(\Sigma)$ in $\mathcal{K}(\mathfrak{H})$ such that $\sigma(T + K) = \Sigma$.*