

Positively Invariant Regions for Systems of Nonlinear Diffusion Equations

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1. Introduction. For a long time, the maximum principle has been used to find positively invariant regions for diffusion equations. From the existence of such regions, one obtains *a priori* bounds on solutions, and in special instances, stability results. In recent papers (*e.g.* [5], [12], [14] and [15]), the maximum principle has been reformulated in terms of comparison functions, and this point of view is adopted here. The first aim of this paper is to sharpen somewhat the basic result of [15] and to add a result on boundedness of derivatives following the work of the first author in [5]. This result on derivatives shows that the ω -limit set of the positively invariant set is compact and makes applicable the theory of flows on compact spaces.

As pointed out in [15], the hypotheses required for the basic result are quite restrictive. In particular, one can have a globally asymptotically stable rest state without having a positively invariant region in the sense intended here. However this essential weakness of the approach does not prevent a reasonably large collection of applications, and this our second aim which we point out here. In another report, [6], a relation between positively invariant regions and traveling wave solutions is discussed. This is relevant for the FitzHugh-Nagumo and the Hodgkin-Huxley equations (see also [4]).

2. The equations. Let \mathfrak{U} and \mathfrak{V} be open sets in \mathbf{R}^m and \mathbf{R}^n respectively, and let S be a positive definite $m \times m$ symmetric matrix-valued function of $x \in \mathfrak{U}$. Then at each point x_0 of \mathfrak{U} , $S(x_0)$ can be written as $\sum \lambda_k^2 \xi_k^i \xi_k^i$, where λ_k^2 is the k th eigenvalue of $S(x_0)$ and ξ_k^i is the i th component of the k th eigenvector.

Given a smooth function $v : \mathfrak{U} \rightarrow \mathfrak{V}$, define $\Delta v(x_0)$ to be $\sum \lambda_k^2 d^2 v(\xi_k)$, where $d^2 v$ denotes the second derivative of v at x_0 .

Now let D and M^j ($j = 1, 2, \dots, m$) be $n \times n$ matrix-valued functions on $\mathfrak{U} \times \mathfrak{V}$ such that for all $(x, v) \in \mathfrak{U} \times \mathfrak{V}$, $D(x, v)$ has real non-negative eigenvalues. Finally, let f be a function from \mathfrak{V} into \mathbf{R}^n .

We consider those functions $v(x, t)$ defined on $\mathfrak{U} \times \mathbf{R}^1 \rightarrow \mathfrak{V}$ which satisfy