

# An Example Concerning Approximate Differentiation

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The purpose of this note is to resolve negatively the following question, which was posed in [1, 3.1.17]:

**Question:** Let  $m$  and  $n$  be positive integers; let  $A \subset U \subset \mathbf{R}^m$ , with  $U$  open; and let  $F : U \rightarrow \mathbf{R}^n$  be a function of class  $k$ . Suppose that for each  $a \in A$ ,

$$(1) \quad \text{ap} \limsup_{x \rightarrow a} \frac{\|D^k F(x) - D^k F(a)\|}{|x - a|} < \infty.$$

Can one conclude that for each  $\epsilon > 0$  there is a map  $G : \mathbf{R}^m \rightarrow \mathbf{R}^n$  of class  $k + 1$  such that  $\mathcal{L}^m(A \sim \{x : F(x) = G(x)\}) < \epsilon$ ?

Here our conventions are those of [1]. In particular, “ $F$  is of class  $k$ ” means that the  $k$ th differential of  $F$ ,  $D^k F$ , exists and is continuous; “ap lim sup” denotes the approximate limit superior; and  $\mathcal{L}^m$  is Lebesgue measure on  $\mathbf{R}^m$ .

We will show that whenever  $k \geq 1$  this question must be answered negatively, by proving the following results:

**Theorem:** Let  $\delta$  be a real number satisfying  $0 < \delta < 1$ . There is a function  $f : \mathbf{R} \rightarrow \mathbf{R}$  of class 1 such that

- (2.1)  $f'$  satisfies a Hölder condition with exponent  $\delta$ , and  $f'(x) \geq 0$  for every  $x \in \mathbf{R}$ ;
- (2.2)  $f'$  is approximately differentiable at almost every point of  $\mathbf{R}$ ;
- (2.3) for every function  $g : \mathbf{R} \rightarrow \mathbf{R}$  of class 2,  $\mathcal{L}^1\{x : f(x) = g(x)\} = 0$ .

**Corollary:** Let  $\delta$  be a real number satisfying  $0 < \delta < 1$ . Let  $m$ ,  $n$ , and  $k$  be positive integers. There is a function  $F : \mathbf{R}^m \rightarrow \mathbf{R}^n$  of class  $k$  such that

- (3.1)  $D^k F$  satisfies a Hölder condition with exponent  $\delta$ ;
- (3.2)  $D^k F$  is approximately differentiable at almost every point of  $\mathbf{R}^m$ ;
- (3.3) for every function  $G : \mathbf{R}^m \rightarrow \mathbf{R}^n$  of class  $k + 1$ ,  $\mathcal{L}^m\{x : F(x) = G(x)\} = 0$ .

The motivation for the above question was two-fold: one knows that the corresponding question with “ap lim sup” replaced by “lim sup” in (1) has an affirmative answer [2; also 1, 3.1.15]; and one knows that when  $k = 0$  the question has an affirmative answer [1, 3.1.16]. Thus, if  $f$  is as described in the theorem, then for each  $\epsilon > 0$  there is a function  $g_\epsilon$  of class 1 satisfying