

# *Finitely Generated, Noetherian, and Artinian Banach Modules*

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**1. Introduction.** In this paper we begin a study of the structure of Banach modules which satisfy various algebraic conditions. We will say that  $X$  is a *Banach module* over  $A$ , or a *Banach  $A$ -module*, if  $A$  is a Banach algebra,  $X$  is a Banach space and a left  $A$ -module, and if the module multiplication is jointly continuous. The two most important examples of Banach  $A$ -modules are, first,  $A$  itself, and, second, a Banach space  $X$  on which  $A$  is a uniformly closed algebra of bounded operators. In the first case the (closed) submodules are the (closed) left ideals of  $A$ ; and in the second case the (closed) submodules are the (closed)  $A$ -invariant linear subspaces of  $X$ , and the (continuous) module homomorphisms are the (continuous) linear operators on  $X$  which commute with the operators in  $A$ . The questions we study in this paper are largely motivated by these two examples. In most cases the applications to ideal theory and to operator algebras are apparent, but we do give several illustrations, in Section 4, of applications which might not be immediately obvious. Section 4 also contains some examples of reasonably complicated Noetherian Banach modules.

In Section 2, we study finitely generated Banach modules. This study includes, as a special case, the operator algebras of finite strict multiplicity studied by Herrero and Lambert [11] [12] [16]. This section includes generalizations and extensions of some of their results, and of earlier results by many authors on strictly cyclic operator algebras. For instance, in Theorem (2.3), we show that every module homomorphism from a finitely generated Banach module to an arbitrary Banach module is continuous, and in Theorem (2.5), we characterize separating strictly cyclic vectors. Section 2 also contains a variety of new results about maximal submodules of finitely generated modules.

The difference between considering  $X$  as a Banach module over  $A$ , and just considering  $A$  as an algebra of operators on  $X$ , is that we no longer need to assume that all the elements of  $A$  act nontrivially on  $X$  or that  $A$  is closed in the uniform topology on  $X$ . While this might seem like only a minor increase in generality, it permits us to effectively use submodules and quotient modules to study  $X$ . Another obvious advantage of the module point of view is that it