

# Nuclear $C^*$ -algebras and Injectivity: The General Case

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In a previous article [3], the authors used Connes' fundamental paper "Classification of injective factors" [4] to prove that a separable  $C^*$ -algebra  $A$  is nuclear if and only if the enveloping von Neumann algebra  $A^{**}$  is injective (see definition below). In this note we will prove that the corresponding result for general  $C^*$ -algebras follows from the separable case. As a consequence, we show that quotients and extensions of arbitrary nuclear  $C^*$ -algebras must again be nuclear.

Given a Hilbert space  $H$ , we let  $\mathfrak{B}(H)$  be the bounded linear operators on  $H$ . An operator system  $N$  on  $H$  is a self-adjoint linear subspace of  $\mathfrak{B}(H)$  that contains the identity operator 1. We denote its self-adjoint elements by  $N_h$ . If  $n = \dim H < \infty$ ,  $\mathfrak{B}(H)$  may be identified with the  $n \times n$  matrices  $M_n$ , and we then say that  $N$  is a matrix system. A von Neumann algebra  $R$  is injective if given operator systems  $N_1 \subseteq N_2 \subseteq \mathfrak{B}(H)$ , any completely positive map  $\varphi : N_1 \rightarrow R$  extends to a completely positive map  $\psi : N_2 \rightarrow R$ .

For integers  $n, k$  with  $0 \leq k, n$ , we define the matrix system  $N_{n,k} \subseteq M_{2n+k}$  to be the hyperplane

$$N_{n,k} = \{ \alpha \in M_{2n+k} : \sum_{i \leq n} \alpha_{ii} = \sum_{i > n+k} \alpha_{ii} \}.$$

We will need the following refinement of [1, Th. 3.4]. We note that it can be shown that one need only consider the spaces  $N_{n,0}$ .

**Lemma 1.** *If  $R$  is a von Neumann algebra, then the following are equivalent:*

- (1)  $R$  is injective.
- (2) For all  $n, k$  with  $0 \leq k, n$ , any completely positive map  $\varphi : N_{n,k} \rightarrow R$  may be extended to  $M_{2n+k}$ .

*Proof.* (1)  $\Rightarrow$  (2) is immediate. Conversely let us assume that  $R \subseteq \mathfrak{B}(H)$  satisfies (2). We recall from [1, Th. 3.4] that  $R$  is injective if and only if it has the following "relative interpolation property" (we use the convenient notation of Connes [4]): given  $\sigma \in (M_m)_h$  and  $s \in M_m(R)_h$ , if there exists an element  $b \in \mathfrak{B}(H)_h$  with  $\sigma \otimes b = [\sigma_{ii}b] \leq s$ , then there exists an element  $r \in R_h$  with  $\sigma \otimes r = [\sigma_{ii}r] \leq s$ .