

On the Theory of Compact Operators in Von Neumann Algebras I

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Introduction. The aim of this work is to inquire to what extent some chapters of the theory of compact operators can be extended to von Neumann algebras. In this first article we succeed to carry over to von Neumann algebras a major part of the classical theories of semi-Fredholm operators and of the essential spectra. A foundation for this extension was laid by the works of Sonis [15] and Breuer [3], [4].

In our first preparatory section we shall show that Sonis' definition of compact operators in a von Neumann algebra with Segal measure is viable for general von Neumann algebras too and then we shall obtain most of the classical characterizations of compact operators.

In the second section we shall define the semi-Fredholm operators. Using the notion of relatively finite nullity we shall characterize them in terms of left (or right) invertibility modulo the compact operators and extend great part of their classical theory.

In the third section we shall apply the results of the previous one to define five essential spectra and obtain most of their classical properties. We shall find however that Weyl's theorem [16] on the essential spectrum of a selfadjoint operator doesn't hold in continuous von Neumann algebras. This will be shown through an example which at the same time will prove that a theorem by Ovchinnikov [13] about the numerability of the set of eigenvalues of a compact operator cannot be extended to von Neumann algebras without Segal measure.

In a next paper we shall introduce the notion of weak convergence of vectors relative to a von Neumann algebra and through it we will extend Hilbert's definition of compact operators and a characterization of the left essential spectrum due to Wolf.

We wish to thank M. Sonis for many helpful discussions.

1. Compact Operators in von Neumann Algebras. Let H be a Hilbert space, $L(H)$ be the algebra of all bounded linear operators on H , $\mathfrak{A} \subset L(H)$ be a von Neumann algebra, and let $\mathcal{P}(\mathfrak{A})$ be the set of all the projections of \mathfrak{A} .