

Applications of the Hessian operator in a Riemannian manifold

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Introduction. In this paper we first derive a formula (see Theorem 1) for the integrals of the invariants of the Hessian of a function on a Riemannian manifold. We then use this formula to obtain several results. The main one is a generalization, to the case of compact manifolds with boundary, of the well-known theorem of Lichnérowicz and Obata on the first eigenvalue of the Laplace operator (Theorem 4). We also give a simple proof of the theorem, due to Aleksandrov, that a compact hypersurface of constant mean curvature which bounds a compact domain in \mathbf{R}^n is a sphere (Theorem 5). After generalizing an integral formula of Bernstein, Heinz and Chern (Theorem 6) we then extend a result of Flanders on functions with negative second invariant (Theorem 7). We also derive some integral formulas (Theorem 3) as well as several other results.

The proof of Theorem 1 is in Section 2 while the applications are in Section 3.

1. Preliminaries. To begin we shall quickly review Cartan's method of moving frames. For more details see [8]. (Note that some of our sign choices differ from those in [8]).

Conventions and Notation. Throughout this paper we shall use M and N to denote smooth connected, oriented Riemannian manifolds of dimensions n and $n - 1$ (respectively). We allow M to have a boundary. We shall further follow the index convention that upper case indices such as A, B, C, D will range from 1 to n while lower case indices such as i, j, k, l will range from 1 to $n - 1$. Also we shall follow the summation convention in which repeated indices are summed over their ranges. Finally, we shall indicate the end of a proof by the symbol ■.

Let e_1, \dots, e_n be a smooth orthonormal frame field defined on an open set U of M . The dual 1-forms, $\omega_1, \dots, \omega_n$, defined by the rule $\omega_A(e_B) = \delta_{AB}$, form the dual coframe field and satisfy Cartan's structure equations: