

Differentiation, Integration, and Lebesgue Area

T. NISHIURA & J. C. BRECKENRIDGE

In [8] Goffman and Liu defined a class of maps, possibly discontinuous, for which a Lebesgue area can be defined. Connections between classes of Sobolev maps and the integral formula for area were subsequently shown by Goffman and Ziemer [10] to be related to the maps of Goffman and Liu. In a recent paper by Goffman and Liu [9] the following regularity condition in a class of Sobolev maps was defined:

A map $f : X \rightarrow \mathbf{R}^n$, where $X \subset \mathbf{R}^n$, is said to be *regular at* $x \in X$ if

$$\operatorname{ess\,lim\,inf}_{\delta \rightarrow 0^+} \frac{L_n[f(\partial I_\delta(x))]}{L_n[I_\delta(x)]} = 0.$$

Here, $I_\delta(x)$ is the oriented n -cube with diameter δ and center x , and L_n is Lebesgue n -dimensional measure.

It was shown in [9] that regularity at almost every point of X implies the equality of Lebesgue area and the classical integral formula for a certain class of Sobolev maps.

Regularity as defined above is expressed in terms of the derivative of the interval function $L_n[f(\partial I)]$. In the present paper we consider the derivatives of two other interval functions related to Lebesgue area. (See Sections 4 and 8 below.) The first interval function is more suited for continuous maps and the second is related to the interval function $L_n[f(\partial I)]$ above. A class of maps is defined in which the difference between the Lebesgue area and the area integral is expressible in terms of essential bounds on these derivatives. Consequently, when the essential bound is zero, we have coincidence of Lebesgue area and the integral formula.

The main results of the present paper are Theorems 3.1, 4.2, and Sections 7 and 8 in which a development of a lower area and signed areas for the $(n - 1)$ -continuous maps of Goffman and Liu is given. Theorem 3.1 states that for flat continuous maps in our class, the area measure and the measure induced by the integral of the formal Jacobian agree on the "light" part of the map. Theorem 4.2 is a characterization of continuous flat maps in our class for which the Lebesgue area and the integral formula agree for each open subset of X . In section 7, lower semi-continuous multiplicity functions are shown to exist for