

Products of Dilatations

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Let V be a right vector space over a division ring F . An invertible linear transformation $A: V \rightarrow V$ is called a *dilatation* if there exists a direct decomposition $V = H \oplus L$ into A -invariant subspaces such that the restriction of A to H is the identity map, the restriction of A to L is not the identity and $\dim L = 1$. This definition is equivalent to the definition of dilatations in [1, p. 3-4]. The subspaces H and L are uniquely determined by A and we shall write $H = H_A$, $L = L_A$ when we wish to indicate this dependence. Note that the inverse of a dilatation is also a dilatation.

When $\dim V = n$ is finite we shall identify $\text{Aut}(V)$ with $GL_n(F)$. In that case an invertible matrix A is a dilatation if it is similar over $GL_n(F)$ to a diagonal matrix whose diagonal entries are $1, 1, \dots, 1, \lambda$ where $\lambda \neq 1$. If $\dim V \geq 1$ and if F has at least 3 elements then $\text{Aut}(V)$ contains dilatations. If F is the field of two elements then there are no dilatations. Our objective is to prove the following

Theorem. *If F has at least four elements then every $A \in GL_n(F)$ is a product of at most n dilatations and this bound is best possible.*

Proof. We shall proceed by induction on n . If $n = 0$ or 1 the assertion is obvious.

Case $n = 2$. If A is diagonalizable the assertion is trivial. Otherwise we may assume that A is in the canonical form

$$A = \begin{pmatrix} 0 & \alpha \\ 1 & \beta \end{pmatrix}.$$

Since F has at least four elements we can choose $\lambda \in F$ distinct from $0, 1, -\alpha^{-1}$. Let

$$\mu = (1 - \beta)\alpha^{-1} - \lambda.$$

We have

$$\begin{pmatrix} \lambda & 0 \\ \mu & 1 \end{pmatrix} A = \begin{pmatrix} \lambda & 0 \\ \mu & 1 \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ 1 & \beta \end{pmatrix} = \begin{pmatrix} 0 & \lambda\alpha \\ 1 & \mu\alpha + \beta \end{pmatrix},$$