

# The Spectrum of a Normal Operator and the Problem of Filling in Holes

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**1. Introduction.** If  $T$  is a bounded operator on a Hilbert space, its spectrum will be denoted by  $\sigma(T)$ . Throughout this paper  $N$  will denote a nonreductive normal operator on a separable Hilbert space  $\mathfrak{K}$  and  $\mathfrak{S}(N)$  will denote the set of all non-normal subnormal operators,  $S$ , for which  $N$  is the minimal normal extension. In general, the notation of this paper is consistent with that of [5].

If  $S$  is a subnormal operator on  $\mathfrak{K}$  and  $N$  is its minimal normal extension on  $\mathfrak{K}$ , then P. Halmos [7] showed that  $\sigma(S)$  contains  $\sigma(N)$ . J. Bram [1] improved this result by showing that the spectrum of  $S$  is obtained by taking the union of  $\sigma(N)$  with some of the bounded components of the complement of  $\sigma(N)$ . (These components will be referred to as the holes of  $\sigma(N)$ .)

J. Conway and R. Olin [5] consider the problem of which holes can be filled in by the operators in  $\mathfrak{S}(N)$ . (For a history of this subject we refer the reader to section nine of that paper.) They showed [Theorem 9.2] that

$$\bigcup_{S \in \mathfrak{S}(N)} [\sigma(S) \setminus \sigma(N)] = \tilde{K} \setminus \sigma(N)$$

where  $\mu$  is a scalar spectral measure for  $N$  and  $\tilde{K}$  is the set used by  $D.$  Sarason [9] to describe the weak-star closure in  $L^\infty(\mu)$  of the polynomials. They then asked [5, Question 10.2] the following: Given an arbitrary collection of components of  $\tilde{K} \setminus \sigma(N)$ , is there an  $S$  in  $\mathfrak{S}(N)$  such that  $\sigma(S)$  is  $\sigma(N)$  together with this collection of components?

The purpose of this paper is to show that this is not always possible. Moreover, we shall give necessary and sufficient conditions that a subnormal operator  $S \in \mathfrak{S}(N)$  fills one hole and not another. It will be obvious how to generalize this theorem to an arbitrary collection of holes and we leave the details to the reader.

**2. The theorem and an example.** For a compact set  $K$  in the plane,  $R(K)$  denotes the uniform closure of the rational functions with poles off  $K$ . If  $\mu$  is a measure with support contained in  $K$ , then  $H^\infty(\mu)$  denotes the weak-star closure of  $R(K)$  in  $L^\infty(\mu)$ . (Note that  $H^\infty(\mu)$  is dependent upon  $K$  as well as  $\mu$ .)

As before,  $N$  is a normal operator on a separable Hilbert space  $\mathfrak{K}$  and  $\mu$  is a scalar spectral measure for  $N$ .