

Dentability, Smoothability and Stronger Properties in Banach Spaces

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1. Introduction. In the past several years remarkable progress has been made toward understanding the geometrical aspects of the Radon–Nikodym property in Banach spaces. (See, for example [4], [9], [10] and [13]. The bibliography of [5] is also an excellent source.) Some of the central results concern the plentitude of extreme points in closed bounded convex sets [10], [16]. The well known duality between extreme points and support functionals suggests that there should be strong connections between the Radon–Nikodym property in a Banach space and various smoothness conditions on the dual or pre-dual of X . It is known that if X has a Frechet differentiable norm or a very smooth norm or a norm with property V [21] then X^* has the Radon–Nikodym property. A deeper result is that X is an Asplund space iff X^* has the Radon–Nikodym property [1], [20].

However, some interesting questions still persist. For example, it is not known whether every Asplund space has an equivalent Frechet differentiable norm—although every known example of a space having either property has both.

In the present note we define a smoothness condition which is dual to dentability. The proof of the duality is based on generalizing some inequalities of Lindenstrauss [12] which we have discussed elsewhere [21]. Section 2 contains a proof of the duality. An immediate corollary is that if the unit ball of X is dentable then every bounded subset of X^* is *smoothable*. The notion of smoothability was introduced by M. Edelstein in [6] and investigated more fully by Kemp in [11]. Our corollary is a slight generalization of a result of Kemp.

Smoothability is expressed in terms of separating sets by closed balls. A related stronger condition is that every closed bounded convex set be the intersection of closed balls. This property was first studied by Mazur [14] and later by R. Phelps [17] who called it property \mathcal{S} . It is immediate from Phelps' results that if the norm in X is Frechet differentiable then X has property \mathcal{S} . V. Zizler [24] proved that if X is an Asplund space with a “directionally lower semi-continuous” duality mapping, then X has property \mathcal{S} .

In Section 3 we give a characterization of property \mathcal{S} , for smooth spaces in terms of the density of certain subsets of X^* which is, in turn, equivalent to