

Smoothness of Bessel Potentials and Lipschitz Functions

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1. The differentiability properties of functions in the Bessel potential space L_{α}^p have been studied extensively. If $p < n/\alpha$, the differentiability results are in terms of the metric in L^r , $1/r \geq (1/p) - (\alpha/n)$ ([1, 4]), with the usual limiting result for $p = n/\alpha$. However, if $p > n/\alpha$, much better differentiability results can be obtained. For k a non-negative integer, let $\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1} f)(x)$, where $(\Delta_h f)(x) = f(x-h) - f(x)$. If $f \in L_{\alpha}^p$, $p > n/\alpha$, k is the first integer $\geq \alpha$, and $M_{\alpha} f(x) = \sup_h |\Delta_h^k f(x)|/|h|^{\alpha}$, then the principal result in [3] asserts that $\|M_{\alpha} f\|_p \leq c \|f\|_{L_{\alpha}^p}$, where c is independent of f . With this result it is easy to obtain differentiability properties of f . The smoothness relation $\Delta_h^k f(x) = O(|h|^{\beta})$ as $h \rightarrow 0$ is also studied in [3], and it is shown that under certain conditions this holds except for a set whose Hausdorff measure or Bessel capacity of a certain index is zero.

In this paper we wish to extend these results to fractional differences

$$\Delta_h^{\alpha} f(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x - jh), \quad \alpha > 0,$$

and study the smoothness condition $\Delta_h^{\alpha} f(x) = O(|h|^{\beta})$, $\alpha > \beta$, for Bessel potential functions, or, more generally, Lipschitz functions. There are interesting relations between $\Delta_h^{\alpha} f$ and the fractional derivative of order α , and for this the reader is referred to a paper by Butzer and Westphal [2].

2. For a general reference for the Bessel potential space $L_{\alpha}^p(\mathbb{R}^n)$ and the Lipschitz space $\Lambda_{\alpha}^p(\mathbb{R}^n)$ the reader should consult [7] or [8]. For $\alpha > 0$, L_{α}^p is the Banach space of all functions f such that $f = G_{\alpha} * g$, $g \in L^p$, where $\hat{G}_{\alpha}(x) = (1 + 4\pi^2 |x|^2)^{-\alpha/2}$ with $\|f\|_{L_{\alpha}^p} = \|g\|_p$. Associated with L_{α}^p is the Bessel capacity $B_{\alpha p}$ which is defined for $E \subset \mathbb{R}^n$ by

$$B_{\alpha p}(E) = \inf \{ \|g\|_p^p : g \in L^p, g \geq 0, \quad G_{\alpha} * g \geq 1 \quad \text{on} \quad E \}.$$

Sets of $B_{\alpha p}$ -capacity zero refine sets of Lebesgue measure zero; in fact, if $p > 1$, $\alpha p < n$, then $B_{\alpha p}(E) = 0$ implies that $H^{n-\alpha p+\epsilon}(E) = 0$ for every $\epsilon > 0$, where H^r is Hausdorff measure of dimension r [5].