

A Luzin Type Property of Sobolev Functions

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1. Introduction. A classical theorem of Luzin states that if a measurable function f (in the sense of Lebesgue) is defined on a bounded domain of R^n , then given $\epsilon > 0$ a compact set F can be obtained by removing a set of measure smaller than ϵ from the domain such that when considered as a function defined on F , f is continuous. When integrability is taken into consideration, Luzin's theorem can be strengthened to yield:

If $f \in L_p(G)$, $1 \leq p < +\infty$, then given $\epsilon > 0$ a compact subset F of G and a continuous function g on G exist such that

$$1) \mu(G \setminus F) < \epsilon;$$

$$2) f = g \text{ on } F;$$

and

$$3) \|f - g\|_p < \epsilon,$$

where, and henceforth, $\mu(\cdot)$ is the Lebesgue measure in R^n , and $\|\cdot\|_p$ is the L_p -norm with respect to $\mu(\cdot)$.

This follows easily from a simple fact stated as Lemma 1 in §3. One may ask whether properties of this nature hold for function spaces endowed with richer structures. Our concern in this paper is to establish the appropriate Luzin theorem for functions in Sobolev spaces. This may be considered as a sequel to the results of Michael [5] and Goffman [3] in answering a conjecture of Goffman [2].

As usual, we use $W_p^{(l)}(G)$ to denote the Sobolev space of equivalence classes of measurable functions all of whose generalized derivatives up to order l are functions belonging to $L_p(G)$, where l is a positive integer and $1 \leq p < +\infty$. $W_p^{(l)}(G)$ is endowed with a norm $\|\cdot\|_p^{(l)}$ defined by

$$\|f\|_p^{(l)} = \sum \|f_J\|_p,$$

where the summation is taken over all $J = (j_1, j_2, \dots, j_n)$ with $j_1 \geq 0, \dots, j_n \geq 0$ and $|J| = j_1 + \dots + j_n \leq l$; f_J is understood to be $\partial^{|J|} f / \partial^{j_1} x_1 \dots \partial^{j_n} x_n$ in the generalized sense. It is well-known that $W_p^{(l)}(G)$ is a Banach space. The closure in $W_p^{(l)}(G)$ of the space of all C^l -functions with compact support in G is denoted by $\dot{W}_p^{(l)}(G)$. $\dot{W}_p^{(l)}(G)$ is a Banach space with norm inherited from $W_p^{(l)}(G)$.