

Prime Divisors and Saturated Chains

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Introduction. Given a noetherian domain R with a nonzero prime ideal M , let $P \neq MR[x]$ be a prime ideal lying over M in the polynomial ring $R[x]$. A maximal chain of prime ideals in R_M of length s gives rise to a maximal chain in $R[x]_P$ of length $s + 1$; but an example of Nagata shows that there may exist maximal chains in $R[x]_P$ not of length $s + 1$ for such an s . However, if we go to the polynomial ring over $R[x]$ and to primes lying over P , this anomalous behavior disappears [12]. In the present paper we relate these two phenomena to certain sets of discrete valuation rings dominating R_M —valuation rings extending the classical notion of “prime divisors in function fields” [13]. Our results consist largely of generalizations and simplifications of certain of the principal results of [9] and [12], and were inspired primarily by the preprint of [9].

Throughout this paper, M and R will be as above, and K will be the quotient field of R . If $A \subset B$ are domains with Q prime in B and $P = Q \cap A$, then $t(B/A)$ and $t(Q/P)$ will denote, respectively, the transcendence degrees of B over A , and B/Q over A/P . By \bar{A} we mean the integral closure of A .

1. Prime divisors. The central objects of our study will be those discrete valuation rings centered on M , which can be obtained by localizing an integral extension of a finitely generated extension of R . We call such valuation rings *prime divisors* (of the local domain R_M); the set of all such will be denoted by $D(R, M)$. If L is an extension field of K , then $D(R, M, L)$ will be the subset of $D(R, M)$ consisting of those members of $D(R, M)$ having quotient field L .

If \bar{R} contains a height 1 prime \bar{M} lying over M , then obviously $\bar{R}_{\bar{M}} \in D(R, M)$ and $t(\bar{M}/M) = 0$. The crucial observation of this paper is that the converse is true:

1.1. Lemma. *If $(V, N) \in D(R, M)$ with $t(N/M) = 0$, then $ht(N \cap \bar{R}) = 1$.*

Proof. Let $R \subset R[d_1, \dots, d_r] \subset A \subset V$ with A integral over $R[d_1, \dots, d_r]$ and $V = A_{(N \cap A)}$. We may assume that $A = \bar{A}$ so that if B is the integral closure of R in A , then $B = \bar{B}$. Now $t(N \cap A/N \cap B) = 0$ and $ht(N \cap A) = 1$ together imply that $N \cap A$ is isolated over $N \cap B$. By the Peskine–Evans formulation of Zariski’s main theorem on birational correspondences [3], we have $B_{N \cap B} = A_{N \cap A}$ so that $ht(N \cap B) = 1$. Since B is integral over \bar{R} , by the Going Down Theorem $ht(N \cap \bar{R}) = 1$.