

Measurability in a Banach Space

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Introduction. Several σ -algebras can be defined in a nonseparable Banach space X . For example, each topology on X (weak*, weak, norm) has associated with it Baire sets, Borel sets, universally measurable sets. The relation between some of these σ -algebras is investigated. We also consider the question of when a scalarly measurable function φ with values in X is weakly equivalent to a Bochner measurable function. It turns out that this is true for all scalarly measurable functions φ if and only if the weak topology on X has a certain property, known in the literature as *measure-compactness*. If the weak topology for X is Lindelof or paracompact (in particular if X is a subspace of a weakly compactly generated space), then it is measure-compact. If the weak topology for X is not realcompact, then it is not measure-compact.

Section 1 concerns Borel sets in X . The weak and norm Borel sets coincide if X admits a Kadec norm (in particular if X is locally uniformly convexifiable). The weak and norm universally measurable sets always coincide. A conjugate space X^* has the Radon-Nikodym property if and only if the weak* and norm universally measurable sets coincide.

Section 2 concerns Baire sets in X . The σ -algebra of weak-Baire sets is the σ -algebra generated by the continuous linear functionals. Thus a function φ from a measure space $(\Omega, \mathcal{F}, \mu)$ into X is scalarly measurable if and only if φ is measurable from \mathcal{F} to the weak-Baire sets of X .

The third section asks what kind of subset of X^{**} (in the weak* topology) the canonical image of X is. If X is weakly compactly generated, or if $X = l^1(\Gamma)$, then X is a weak*-Borel set in X^{**} . Any Banach space X is weak*-universally-measurable in X^{**} (a result of Tortrat),

Section 4 reviews material concerning measures in a topological space, such as measure-compact topological spaces and tight measures. Section 5 concerns scalar and Bochner measurability of functions $\varphi : \Omega \rightarrow X$ from a measure space $(\Omega, \mathcal{F}, \mu)$ to a Banach space X . The scalarly measurable function φ is weakly equivalent to a Bochner measurable function if and only if the image measure $\varphi(\mu)$ on X is tight (according to the weak topology on X). Thus, every scalarly measurable function φ in X is weakly equivalent to a Bochner measurable function if and only if every measure on the weak-Baire sets of X is tight, *i.e.* the weak topology is “strongly measure-compact”; this happens if and only if the weak topology is measure-compact.