

On a Best Estimator for the Class \mathfrak{M}^r Using only Function Values

CHARLES A. MICCHELLI & ALLAN PINKUS

1. Introduction. Let $BV[0, 1]$ denote the set of all real-valued functions of bounded variation on $[0, 1]$. For $r \geq 2$, we define the class of r -fold integrals of elements in $BV[0, 1]$ by

$$\mathfrak{M}^r = \mathfrak{M}^r[0, 1] = \left\{ f : f(x) = \sum_{i=1}^r a_i x^{i-1} + \frac{1}{(r-1)!} \int_0^1 (x-t)_+^{r-1} d\lambda_r(t), \right. \\ \left. \lambda_r \in BV[0, 1], (a_1, \dots, a_r) \in \mathbf{R}^r \right\}.$$

Fix $\mathbf{x} = (x_1, \dots, x_n)$, $0 < x_1 < \dots < x_n < 1$, $n \geq r$, and let \mathbf{f} denote the vector $(f(x_1), \dots, f(x_n))$. In this paper, we are concerned with best methods of estimating a function $f \in \mathfrak{M}^r$, given its function values \mathbf{f} . Without any additional prior information on f , arbitrarily large errors in estimating f may be encountered. It is the purpose of this paper to describe an optimal estimator for f when a bound on the total variation $\|\lambda_r\|$ of λ_r on $[0, 1]$ is available.

The specific problem we will treat is formulated as follows:

Any mapping T (not necessarily linear) from \mathbf{R}^n into \mathfrak{M}^r determines an estimate $Sf = T\mathbf{f}$ for f . The (relative) error for this estimate S , given only that $f \in \mathfrak{M}^r$, is defined as

$$E(\mathbf{x}; S) = \sup_{f \in \mathfrak{M}^r} \frac{\|f - Sf\|_1}{\|\lambda_r\|},$$

where $\|\cdot\|_1 = L^1$ -norm on $[0, 1]$. We call S_0 an optimal estimator provided that

$$E(\mathbf{x}) = \inf_s E(\mathbf{x}; S) = E(\mathbf{x}; S_0),$$

where the infimum is taken over all estimators S , as defined above.

Our main result depends on the following lemma which we prove in Section 2. Before stating this lemma, we require the following definitions.

Let $\mathfrak{s}_{r,n} = \mathfrak{s}_{r,n}(\mathbf{x})$ denote the class of spline functions of order r with the n knots $\mathbf{x} = (x_1, \dots, x_n)$, that is,

$$\mathfrak{s}_{r,n} = \left\{ s : s(x) = \sum_{i=1}^r a_i x^{i-1} + \sum_{i=1}^n a_{i+r} (x - x_i)_+^{r-1}, (a_1, \dots, a_{n+r}) \in \mathbf{R}^{n+r} \right\},$$