

# Density Conditions in the $n$ -Capacity

O. MARTIO & J. SARVAS

**1. Introduction.** Let  $C$  be a closed set in  $R^n$ ,  $n \geq 2$ . We compare three density conditions at a point  $x \in C$ :

(1.1) The upper capacity density is zero, *i.e.*  $\lim_{t \rightarrow 0} \varphi(t) = 0$ , where for  $t > 0$

$$\varphi(t) = \varphi(x, C; t) = \text{cap} (B^n(x, 2t), \bar{B}^n(x, t) \cap C)$$

and  $\text{cap}$  refers to the  $n$ -capacity, or the conformal capacity, of a condenser in  $R^n$ .

(1.2) The continuum criterion  $M(x, C) < \infty$ , *i.e.* there is a non-degenerate continuum  $K$  in  $C^c \cup \{x\}$  such that  $x \in K$  and  $M(\Gamma_K) < \infty$ . Here  $M(\Gamma_K)$  denotes the  $n$ -modulus of the path family  $\Gamma_K$  joining  $K$  and  $C$  in  $C^c$ .

(1.3) The (converse) Wiener criterion

$$\int_0 \frac{\varphi(t)^{1/(n-1)}}{t} dt < \infty.$$

The conditions (1.1), (1.2), and (1.3), or rather their converses, have been used to study the regularity of solutions of certain elliptic differential equations and variational extremals at a boundary point  $x$  of a domain  $G$  with  $C = G^c$ , see [6], [3], [1], and [2]. The continuum criterion also seems to be a useful condition to study asymptotic values of analytic and quasiregular mappings at thin boundary points, see [7].

In Section 2 we derive some equivalent definitions for the above conditions; especially, we give for (1.2) an integral representation similar to (1.3). In Section 3 we show that (1.3) implies (1.2) and (1.2) implies (1.1), and we study the failure of the converse statement. In the last section we prove that each of the above conditions is invariant under quasiconformal mappings in the following strong sense: If  $f : C^c \rightarrow R^n$  is quasiconformal and  $x$  satisfies (1.i),  $i = 1, 2$ , or 3, then  $y = \lim_{z \rightarrow x} f(z)$  exists and  $y$  satisfies (1.i) with respect to  $f(C^c)^c$ .

## 2. Preliminaries and equivalent definitions.

**2.1 Notation.** We let  $R^n$  denote the euclidean  $n$ -space with its usual norm. A continuum in  $R^n$  is a compact connected set which contains more than a point.