

# *WC Functions and the Closed Intersection Property*

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In [1] it was shown that if  $X$  is any  $T_3$  space and  $A$  any compact subset of  $WX$  then  $A \cap X$  is closed in  $X$  (*i.e.* that  $X$  has the closed intersection property) and that any Wallman extendible function having a  $T_3$  range is a  $WO$  function (see [4]). In [2] was demonstrated the existence of nonregular Hausdorff spaces having the closed intersection property. In this paper it will be shown that in fact any Wallman extendible function whose range has the closed intersection property must have a closed Wallman extension, and thus must be a  $WC$  function (see [3]).

Recall that for any  $T_1$  space  $X$  the Wallman compactification  $WX$  is the collection  $\{u : u \text{ is an ultrafilter in the lattice of all closed subsets of } X\}$  with the topology generated by  $\{c(A) = \{u \in WX : A \in u\} : A \text{ is a closed subset of } X\}$  as a base for the closed sets. The Wallman compactification  $WX$  is a compact  $T_1$  space and the function  $\varphi_X : X \rightarrow WX$  defined by  $\varphi_X(x) = \{A : A \text{ is a closed subset of } X \text{ and } x \in A\}$  is a dense embedding. A continuous function  $g : WX \rightarrow WY$  is said to be a Wallman extension of  $f : X \rightarrow Y$  if  $\varphi_Y \circ f = g \circ \varphi_X$ . When no ambiguity can result it is common practice to ignore the distinction between a space  $X$  and the subspace  $\varphi_X[X]$  of  $WX$  and to speak of  $X$  as a subspace of  $WX$ . In [3] Harris shows that if a function  $f : X \rightarrow Y$  has a closed Wallman extension (*i.e.* if  $f$  is a  $WC$  function) then the extension of  $f$  is unique. We note also that it is easily shown that if  $A$  is a closed subset of  $X$ , any compact subset of  $WX$  containing  $A$  contains  $c(A)$ .

**Theorem.** *If  $f : X \rightarrow Y$  has a Wallman extension  $f^* : WX \rightarrow WY$  and if  $Y$  has the closed intersection property then  $f$  is a  $WC$  function.*

*Proof.* Consider first a closed subset  $A$  of  $X$ . Since  $f^*$  is continuous and  $c(A)$  is a closed subset of  $WX$ , the image  $f^*[c(A)]$  is a compact subset of  $WY$ , and so  $Y \cap f^*[c(A)]$  is a closed subset of  $Y$  which clearly contains  $f[A]$ . Therefore  $f^*[c(A)]$  contains  $c(c1_Y(f[A]))$ . However  $f^{*-1}[c(c1_Y(f[A]))]$  is a closed subset of  $WX$  which contains  $A$  and therefore  $c(A)$ . Hence  $f^*[c(A)]$  must be equal to  $c(c1_Y(f[A]))$ , a closed set; so the image under  $f^*$  of any basic closed set is closed. Now, for any closed subset  $G$  of  $WX$ , there is some filter  $\mathfrak{F}$  in the lattice closed