

Weighted Shifts and Products of Subnormal Operators

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0. Introduction. An operator T on a Hilbert space \mathcal{H} is called subnormal if there exists a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a normal operator U on \mathcal{K} such that $U\mathcal{H} \subset \mathcal{H}$ and the restriction $U|_{\mathcal{H}} = T$. We say that a collection

$$\{T_1, \dots, T_N\}$$

of commuting subnormal operators on \mathcal{H} has a commuting normal extension if there exist commuting normal operators U_1, \dots, U_N defined on some $\mathcal{K} \supset \mathcal{H}$ with $U_i|_{\mathcal{H}} = T_i$, $i = 1, 2, \dots, N$. T. Ito [3] gave a necessary and sufficient condition for commuting subnormals $\{T_1, \dots, T_N\}$ to have a commuting normal extension by generalizing the positivity condition of Halmos and Bram. Ito's condition, however, is very difficult to apply and in fact, it is not immediately evident from it whether or not there exist commuting subnormals without commuting normal extension. Clearly, if $\{T_1, T_2\}$ has a commuting normal extension, then $T_1T_2 = U_1U_2|_{\mathcal{H}}$ and $(T_1 + T_2) = (U_1 + U_2)|_{\mathcal{H}}$ are subnormal. We give an example below of two commuting subnormals whose product, and also whose sum, are not subnormal, and hence for which a commuting normal extension does not exist. We conclude with a related dilation result that strengthens a theorem of B. Sz-Nagy [8].

Our example is provided by an N -variable analog of weighted shifts. If $\{e_0, e_1, \dots\}$ is an orthonormal basis for \mathcal{H} and $\{a_0, a_1, \dots\}$ is a bounded sequence of positive numbers, the bounded linear map defined by $Te_n = a_ne_{n+1}$ is called a weighted shift. These maps have been a rich source of examples in both operator theory and, since T is unitarily equivalent to multiplication by z on a weighted H^2 -space, also function theory. An excellent introduction to weighted shifts and an extensive bibliography can be found in the recent survey article by A. L. Shields [6]. The subnormal weighted shifts were characterized by J. G. Stampfli [7], and a very different characterization was later given by C. Berger [2] and independently by R. Gellar and L. Wallen [1]. In this paper we extend these characterizations to the N -variable case and as an easy application obtain our main example. We note that by examining when certain elements of the commutant of a subnormal lift to the commutant of