

Inequalities for Integral Bilinear Forms with Applications to Mechanical Systems

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1. Introduction. The results presented in this paper are motivated by the following considerations. If $K(x, t)$ is the Green's function of a one dimensional mechanical system, then the transverse displacement due to a distributed load, $p(x)$, $x \in [0, 1]$, is given by

$$u(x) = \int_0^1 K(x, t)p(t)dt.$$

The potential energy of the system is then given by

$$(1.1) \quad E(K, p) = \int_0^1 \int_0^1 K(x, t)p(x)p(t)dx dt,$$

(see [4]).

The problem to be considered in this paper is that of finding extreme values of this energy when the load p is restricted to be in a given class of functions. The main classes to be considered here will be the classes of *equimeasurable* functions on $[0, 1]$. Two functions, f and g , are said to be *equimeasurable* if for every real number y , the sets $\{x|f(x) \geq y\}$ and $\{x|g(x) \geq y\}$ have the same measure.

Following Hardy, Littlewood and Polya [7] we define the rearrangement \tilde{f} , of a nonnegative, integrable function on $(0, 1)$ in increasing order as follows. Let $M(y)$ denote the measure of the set $\{x|f(x) \leq y\}$ and define \tilde{f} at $x = M(y)$ to be $\tilde{f}(x) = y$. We note that \tilde{f} is an increasing function of x , defined uniquely on $(0, 1)$ except for at most an enumerable set of values of x , *viz.* those corresponding to intervals of constancy of $M(y)$. We may complete the definition of $\tilde{f}(x)$ by agreeing that at a point of discontinuity, we have

$$\tilde{f}(x) = \frac{1}{2}(\tilde{f}(x - 0) + \tilde{f}(x + 0)).$$

We now define the rearrangement of f in symmetrically decreasing order, \hat{f} , as follows:

$$\hat{f}(x) = \tilde{f}(2x), x \in [0, \frac{1}{2}] \quad \text{and} \quad \hat{f}(x) = \tilde{f}(1 - x), x \in [\frac{1}{2}, 1].$$