

Toeplitz Operators with Sectorial Matrix-Valued Symbols

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1. Introduction. It is a classical result that if the real part of a bounded measurable function φ on the unit circle is bounded away from zero, then the associated Toeplitz operator T_φ on the Hardy space H^2 is invertible. An immediate consequence of this fact is that the operator T_φ will be invertible whenever its symbol is sectorial, that is, the closed convex hull of the essential range of φ does not contain zero. Indeed, one has only to observe that such a set can be rotated around the origin into the right half-plane. A local principle of Simonenko (see [5] and the references there) shows further that the operator T_φ is Fredholm whenever φ is locally sectorial. Thus the concept of (locally) sectorial symbol provides sufficient geometric criteria for the regularization of Toeplitz operators with scalar symbols.

In seeking to generalize these results to the matrix case, R. Douglas and H. Widom [2] suggested calling a set of $n \times n$ matrices *sectorial* if every matrix in its closed convex hull is invertible. They then raised the question of whether every compact sectorial set in M_n can be “rotated” into a set of matrices with positive real parts:

Question 1.1. *Let \mathcal{K} be a compact convex set of invertible matrices. Do there exist invertible matrices P and Q such that*

$$PKQ + (PKQ)^* \geq \epsilon I > 0$$

for all $K \in \mathcal{K}$?

In order to see the significance of this question, let $A \in L^\infty \otimes M_n$ and write T_A for the associated Toeplitz operator. Exactly as in the scalar case, if the real part of the matrix function A is strictly positive, then T_A is invertible. If the answer to Question 1.1 were yes, then T_A would be invertible whenever the essential range of its symbol A is sectorial. Further, by Simonenko’s localization technique, it follows that an affirmative answer to Question 1.1 would lead to an affirmative answer to the following question as well:

Question 1.2. *Suppose the essential range of the matrix symbol $A \in L^\infty \otimes M_n$ is sectorial (locally sectorial). Must the associated Toeplitz operator T_A be invertible (respectively Fredholm)?*