

Weakly Dense Subgroups of Banach Spaces

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1. Introduction. In a recent talk, D. J. Newman asked whether the following statement is true for every (real) Banach space X and every proper closed (additive) subgroup G of X :

(*) *There is a non-zero continuous linear functional on X which assumes only integer values on G .*

If (*) were always true, it could be applied to groups in much the same way as the Hahn-Banach Theorem is applied to convex sets, resulting in abstract approximation theorems. The truth of (*) is well-known and not difficult to verify (see proof of Theorem 1) if X is finite-dimensional; it is trivial if the linear span of G is not dense in X .

It is our primary purpose here to show that this statement is false, at least if X is separably infinite-dimensional, for some proper closed subgroup of X (Theorem 5); this includes $X = C[0, 1]$, the case of special interest to Newman. The same conclusion holds for many non-separable X , and we have yet to find such an X for which we can verify (*) for all G . The case in which there is a sublattice (*i.e.*, discrete subgroup) for which (*) fails is of special interest, and arises naturally *en route* to the main result.

The connection between (*) and the title is made by

Theorem 1. *If G is an additive subgroup of a real Banach space $X \neq \{0\}$, then (*) fails to hold if and only if G is weakly dense in X .*

This and other general results will be proved in Section 2. In Section 3 the main results—construction of counter-examples to (*)—are presented. Finally, a related finite-dimensional problem is discussed in Section 4.

Hereinafter the scalar field for all Banach spaces will be assumed to be \mathbb{R} , the field of real numbers. Our results generally have complex analogues which follow immediately if we insist that G be closed under multiplication by i and if we replace the integers \mathbb{Z} by $\mathbb{Z} + i\mathbb{Z}$.

I am indebted to the referee and E. Odell for the present version of the main result (Theorem 5), which is substantially more general than my original theo-