

An Operator Algebra Approach to Partial Differential Equations; Propagation of Singularities and Spectral Theory

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The purpose here is to present a soft analysis approach to two aspects of the theory pseudodifferential and Fourier integral operators. A popular setting in operator theory is an algebra \mathcal{A}_0 of operators which commutes modulo the compact operators. In this paper we study a one parameter group of automorphisms γ_t acting on \mathcal{A}_0 . Naturally the order 0 pseudodifferential operators on a manifold form such an \mathcal{A}_0 while a first order self-adjoint pseudodifferential operator A induces a one parameter group γ_t acting on \mathcal{A}_0 . Thus it is reasonable to try to obtain differential equations results (involving only ‘topological statements about the cotangent sphere bundle of the manifold’ and no geometry) purely from abstract operator theory. We also mention that the higher dimensional Toeplitz operators of Venugopolkrisna lie closely enough to this abstract setting that our work might have implications for them, *e.g.*, a propagation of singularities theorem.

The first part of the paper concerns the celebrated theorem of Hormander which describes the behavior of singularities in the solution to a partial differential equation. Our discussion begins with an easily proved abstract operator algebra theorem, and then goes on to describe how it yields a reasonable version of the standard propagation of singularities theorem. The proof is fairly algebraic and it is amusing that the constructions used look somewhat Galois theoretic. The second part of the paper describes a *canonical* relationship between the spectrum of the operator A and orbits of the ‘Hamiltonian’ flow induced by A . The only thing involved is elementary operator theory, but the results obtained have the flavor of work done by Chazarian, Guillemin-Weinstein and Ralston in partial differential equations.

§ 1. Orbits and the wave front set. The setting is a refinement of a typical situation involving operators on a Hilbert space H . Let \mathcal{A}_0 be a selfadjoint algebra of bounded operators on H which commutes modulo $\mathcal{L}\mathcal{C}$ the ideal of compact operators on H . This algebra will frequently not be closed in the oper-