

# *Reflexive Subspace Lattices in Finite-Dimensional Hilbert Spaces*

K. J. HARRISON & W. E. LONGSTAFF

**1. Introduction.** A closed linear subspace  $M$  of a complex Hilbert space  $\mathcal{H}$  is said to be invariant under a bounded linear operator  $T$  on  $\mathcal{H}$  if  $x \in M$  implies  $Tx \in M$ . Invariance induces a correspondence between sets of subspaces and sets of operators: if  $\mathcal{L}$  is a set of subspaces and  $\mathfrak{A}$  is a set of operators, let  $\text{Alg } \mathcal{L}$  denote the set of all operators that leave invariant each subspace in  $\mathcal{L}$ , and let  $\text{Lat } \mathfrak{A}$  denote the set of subspaces which are invariant under each operator in  $\mathfrak{A}$ . The notation, due to P. R. Halmos, is informative:  $\text{Alg } \mathcal{L}$  is an operator algebra, *i.e.* a subalgebra of  $\mathcal{B}(\mathcal{H})$ , the algebra of all operators, and  $\text{Lat } \mathfrak{A}$  is a subspace lattice, *i.e.* a sublattice of  $\mathfrak{v}(\mathcal{H})$ , the lattice of all subspaces.

It is natural to call a subspace lattice  $\mathcal{L}$  reflexive if  $\mathcal{L} = \text{Lat Alg } \mathcal{L}$ , and to look for reflexive subspace lattices. A necessary condition for reflexivity is not difficult to find [3]: a reflexive subspace lattice is a complete sublattice of  $\mathfrak{v}(\mathcal{H})$  containing the trivial subspaces  $\{0\}$  and  $\mathcal{H}$ . For the remainder of this paper all subspace lattices under consideration are complete and contain the trivial subspaces. Of particular interest are lattice-theoretic conditions which guarantee reflexivity. One of the earliest theorems of this type is due to R. E. Johnson [6]: each distributive subspace lattice in a finite-dimensional space is reflexive. Other results can be found in [3], [5] and [9]. A theorem due to W. E. Longstaff [7] seems to include most of the earlier results: each completely distributive subspace lattice in a space of arbitrary dimension is reflexive.

It is known however that reflexivity is not always an invariant of the lattice-isomorphism type. For example, J. B. Conway [2] has exhibited a pair of lattice-isomorphic subspace lattices only one of which is reflexive. Thus in general extra conditions need to be imposed before reflexivity is assured. One such result is due to W. B. Arveson [1]: each commutative, strongly closed, subspace lattice in a separable Hilbert space is reflexive. A subspace lattice is said to be commutative if the corresponding orthogonal projections commute, and strongly closed if this set of projections is closed in the strong operator topology. Because commutativity of the projections ensures distributivity of the subspace lattice, the condition is algebraic, topological and lattice-theoretic.

In this paper we restrict attention to subspace lattices in finite-dimensional spaces and consider what could be called “small sublattice conditions”. If  $\mathcal{H}$  is