

Operator Algebras with Complemented Invariant Subspace Lattices

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§1. Introduction. The present work is inspired by Rosenthal and Sourour [12] and [13] as well as Feintuch and Rosenthal [5].

An algebra \mathcal{A} of bounded operators on a Hilbert space H is said to be *reductive* if it is strongly closed, contains I and has the property that if $M \subset H$ is a (closed) subspace invariant for every operator in \mathcal{A} , then so is M^\perp . Feintuch and Rosenthal [5] proved that if \mathcal{A} is a reductive algebra and every graph transformation for \mathcal{A} (defined later) is bounded, then \mathcal{A} is self-adjoint. We generalize this result to operator algebras on Banach spaces. First let us fix some notation.

Let X be a Banach space. For (closed) subspaces M and N in X , we write $X = M \dot{+} N$ if $M \cap N = \{0\}$ and X is the algebraic direct sum of M and N . The algebra of all (bounded) operators on X will be denoted by $\mathcal{L}(X)$. For an algebra \mathcal{A} in $\mathcal{L}(X)$, we write \mathcal{A}' for the commutant of \mathcal{A} (i.e., $\mathcal{A}' = \{T \in \mathcal{L}(X) : TA = AT \text{ for all } A \text{ in } \mathcal{A}\}$) and \mathcal{A}'' for the double commutant of \mathcal{A} (i.e., $\mathcal{A}'' = (\mathcal{A}')'$). We also write $\text{lat } \mathcal{A}$ for the collection of those subspaces which are invariant for every operator in \mathcal{A} . For a set \mathcal{S} of subspaces of X , we write $\text{alg } \mathcal{S}$ for the algebra of those operators which leave every subspace in \mathcal{S} invariant. An algebra \mathcal{A} is said to be *reflexive* if $\text{alg lat } \mathcal{A} = \mathcal{A}$.

The strong operator topology of $\mathcal{L}(X)$ is the locally convex topology with a basis of open sets of the form

$$\{T \in \mathcal{L}(X) : \|(T - T_0)x_j\| < \varepsilon, j = 1, \dots, n\}$$

where x_1, \dots, x_n is a finite set in X and $T_0 \in \mathcal{L}(X)$. An operator algebra is said to be *strongly closed* if it is closed in this topology.

An operator algebra \mathcal{A} is said to be *completely reducible* if it is strongly closed, contains I and has the following property: for every subspace M in $\text{lat } \mathcal{A}$, there is a subspace N in $\text{lat } \mathcal{A}$ such that $M \dot{+} N = X$. Obviously reductive algebras are completely reducible. Note that if \mathcal{A} is completely reducible and reflexive, then $\mathcal{A} = \mathcal{A}''$.

An operator defined on a Banach space is said to be completely reducible if the strongly closed algebra generated by it and I is completely reducible. For an operator T , we write $\text{lat } T$ for the lattice of invariant subspaces of T . For