

Positive Operators on the Disk Algebra

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Let A denote the disk algebra equipped with the sup-norm. All operators considered here are bounded linear operators from A to A : Let P denote the set of operators having norm 1 which carry the constant function 1 into itself. It is not hard to show that an operator T is in P iff $T1 = 1$ and T is positive, *i.e.*, $\operatorname{Re} f \geq 0$ implies $\operatorname{Re} Tf \geq 0$. In this paper we study the extremal structure of the convex set P . We are particularly interested in subsets of P of the form $K(F, G) = \{T \in P \mid TF = G\}$, where F and G are members of the closed unit ball U of A . If G is an extreme point of U , then $K(F, G)$ is an extreme subset of P , *i.e.*, $T_1, T_2 \in P$, $c \in (0, 1)$ and $cT_1 + (1 - c)T_2 \in K(F, G)$ implies $T_1, T_2 \in K(F, G)$. Thus, any extreme element of $K(F, G)$ will be an extreme element of P . It will be convenient to consider three classes \mathcal{J}, E_0, E_1 , of extreme elements of U where \mathcal{J} denotes the collection of non-constant inner functions, E_0 denotes the collection of non-constant exposed points of U , and E_1 denotes the collection of non-constant extreme points of U . If F and G are in \mathcal{J} , then the dimension of $K(F, G)$ is finite and can be calculated. (See [2], [7].) In Section 1 we show that if G is in \mathcal{J} and is non-constant, then $K(F, G) \neq \emptyset$ iff $F \in \mathcal{J}$. We also show that, for $i = 0, 1$, if G is in E_i and $K(F, G) \neq \emptyset$, then $F \in E_i$. Furthermore, our methods lead to the proof of a conjecture of Rochberg [7]. In Section 2 we outline a method which can be used to analyze the structure of $K(F, G)$ in cases where F and G are both inner. We also illustrate our method by applying it to the following special cases: F inner and $G = Z$, where Z is the identity on the closed unit disk \bar{D} , and $F = Z^2$ and $G = Z^2$.

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Section 1. $K(F, G) \neq \emptyset$ and $G \in \mathcal{J} (E_i)$ implies $F \in \mathcal{J} (E_i)$. For $f \in A$ we will use Pf to denote the orbit of f under the action of P , *i.e.*, $Pf = \{Tf \mid T \in P\}$. The uniform closure of Pf will be denoted by $clPf$. The notion of orbit was introduced in a similar context by Ryff [9].

Theorem 1.1. *Suppose $F \in U$. If $clPf$ contains a non-constant inner function, then F is inner.*