

An Abstract Kato's Inequality for Generators of Positivity Preserving Semigroups

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§ 1. Introduction. Several years ago, Kato [11] proved a simple but extremely useful inequality:

$$(1) \quad \Delta|u| \geq (\operatorname{sgn} u)\Delta u$$

for u real valued with $u \in L^1_{\text{loc}}$ and $\Delta u \in L^1_{\text{loc}}$ (distribution inequality). (1) is to be interpreted as a distributional inequality although it is proven by a limiting argument starting with nice u 's. The use Kato made of (1) was to prove:

Theorem 1.1. (Kato [11]). *Let $V \in L^2_{\text{loc}}(\mathbb{R}^n)$, $V \geq 0$. Then $-\Delta + V$ is essentially self-adjoint on $C^\infty_0(\mathbb{R}^n)$.*

Kato was motivated in part by a result of Simon [15] who used hypercontractive semigroup methods (see §X.9 and its notes in [12]) to prove the weaker result where $V \in L^2_{\text{loc}}(\mathbb{R}^n)$ is replaced by $L^2(\mathbb{R}^n, \exp(-ax^2)dx)$ for some a . By using the simpler contractive semigroup methods, Semenov [14] (see also Davies [4] and Faris [5]) noted that one can prove the result with $V \in L^2(\mathbb{R}^n, dx)$. There is clearly a relation between the hypercontractive and contractive semigroup methods, but there seems to be little connection between those proofs and Kato's proof. Our main purpose in this note is to show that the methods are related since both depend on the fact that $e^{t\Delta}$ has a positive kernel: this positivity and $\Delta 1 = 0$ lead to the fact that $e^{t\Delta}$ is a contractive semigroup: our point in this note is that (1) is "essentially" equivalent to this positivity.

The link between (1) and positivity preserving semigroups is via the following theorem of Beurling-Deny [2]:

Theorem 1.2. (Beurling-Deny [2]). *Let H be a positive self-adjoint operator on $L^2(M, d\mu)$. Then $\exp(-tH)$ is positivity preserving for all $t > 0$, if and only if $u \in Q(H)$, the form domain of H implies $|u| \in Q(H)$ with $(|u|, H|u|) \leq (u, Hu)$.*