

Structure of the Non-Monotone Free Boundaries in a Filtration Problem

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0. Introduction. In this paper we generalize the results of [5] and [7]. These papers treat a problem in fluid flow through a porous media by transforming the original problem to a variational inequality. We too consider the variational inequality, but by using different techniques we obtain what is a considerably better result.

There is an explanation of the physical problem in [5], so we start by giving the mathematical problem generated by the physical one. Find $u(x, y)$, $\varphi(x)$ for $0 < x < a$ and $0 < y < \varphi(x)$ such that

$$(0.1) \quad H \leq \varphi(0) \text{ and } h \leq \varphi(a), H > h > 0$$

$$(0.2) \quad u(0, y) = H \text{ if } 0 < y < H,$$

$$(0.3) \quad u(a, y) = h \text{ if } 0 < y < h,$$

$$(0.4) \quad u(0, y) = y \text{ if } H \leq y < \varphi(0),$$

$$(0.5) \quad u(a, y) = y \text{ if } h \leq y < \varphi(a),$$

$$(0.6) \quad u(x, \varphi(x)) = \varphi(x) \text{ if } 0 < x < a,$$

$$(0.7) \quad u_{xx} + u_{yy} = 0 \text{ if } 0 < x < a, 0 < y < \varphi(x).$$

$$(0.8) \quad \frac{\partial u}{\partial n}(x, \varphi(x)) = -c\{1 + (\varphi'(x))^2\}^{-1/2} \text{ if } 0 < x < a$$

where n is the unit normal to graph (φ) ,

$$(0.9) \quad u_y(x, 0) = \ell(x) \text{ if } 0 < x < a.$$

Using the transform introduced by Baiocchi in [1], (0.1)–(0.9) is changed into the variational inequality (1.6), (1.7) in Section 1 of this paper. A more detailed explanation of this may also be found in [5].

In Section 1 we deal with the existence and smoothness of $\varphi(x)$. Then Section 2 develops the necessary machinery to estimate $\varphi'(x)$. Section 3 studies the function w_x where w is the solution of (1.7). From this we obtain the sets where $\varphi'(x) \geq 0$ and $\varphi'(x) \leq 0$. Finally, in Section 4 we put all these results together to obtain an estimate on the number of sign changes of $\varphi'(x)$. As a corollary we also prove the main result of [5].