

# *The Methods of Kaczmarz and Row Orthogonalization for Solving Linear Equations and Least Squares Problems in Hilbert Space*

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**1. Introduction.** Let  $H_1$  and  $H_2$  denote real (or complex) Hilbert spaces whose inner-product in both cases we denote by  $\langle \cdot, \cdot \rangle$ . Suppose that  $A: H_1 \rightarrow H_2$  is a bounded linear operator and that  $b \in H_2$ . This paper is primarily concerned with the treatment of two related iterative methods for solving the linear equation

$$(1) \quad Ax = b, x \in H_1.$$

The first is an extension of the method of Kaczmarz [4] which is considered in Section 2. The second is related to the direct matrix method of row orthogonalization (*cf.* [2], [3], [8]). Primarily because it is iterative, the method treated here is significantly more stable and efficient, is well suited for extension to infinite dimensions, and applies to least squares problems as well. This technique is the subject of Section 3. Section 4 contains a short discussion of the generalizations of these two procedures for solving nonlinear algebraic equations. In Section 5 we conclude with a report on some numerical experiments with iterative row orthogonalization.

**2. Kaczmarz method.** In this section, we assume  $b \in R(A)$ , the range of  $A$ , so that a solution of (1) is guaranteed to exist. Letting  $N(A)$  and  $N(A)^\perp$  denote the null space of  $A$  and its orthogonal complement, respectively, the vector  $u$  will represent the unique element of  $N(A)^\perp$  that satisfies  $Au = b$ . Note that the set of solutions of (1) is then given by the linear variety  $u + N(A)$ . For each  $x \in H_1$ , let  $\Delta x = x - u_x$  where  $u_x = u + P_{N(A)} x$ . (For any subspace  $S$  of either  $H_1$  or  $H_2$ ,  $P_S$  will denote the orthogonal projection operator mapping the respective Hilbert space onto  $S$ .) Note that  $\Delta x \in N(A)^\perp$ . For any subset  $X$ , of  $H_1$  or  $H_2$ , Let  $SpX$  and  $\bar{X}$  denote, respectively, the linear span and closure of  $X$ . Let  $\Omega$