

On a Problem of Chen, Willmore, et al

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Let M be a closed orientable C^∞ surface and $f: M \rightarrow E^3$ be a C^k immersion of M into Euclidean three-space, with k taken sufficiently large for what follows. Set

$$(1) \quad \tau(f) = \int_M H^2 * 1,$$

where H is the mean curvature of the immersed surface and $*1$ is the area element of the induced metric on M . Associated with this integral is a variational problem suggested by Chen [4] and Willmore [13], among others: Let M_g be a closed orientable C^∞ surface of genus g and $I^k(M_g, E^3)$ be the set of C^k immersions of M_g into E^3 . Now τ is a function on $I^k(M_g, E^3)$, and we ask for the minimum value of τ on this set if it exists and a characterization of the immersions that assume the minimum. It is known [2] that τ on $\bigcup_{g \geq 0} I^k(M_g, E^3)$ has a minimum value of 4π and that this value is taken on only by a standard sphere in E^3 . The variational problem remains open for surfaces of genus $g \geq 1$.

By computing the first variation of τ one finds that its stationary points are characterized by the partial differential equation

$$(2) \quad \Delta H + 2H^3 - 2HK = 0,$$

where Δ denotes the Laplacian and K the Gauss curvature. Chen has shown that equation (2) is satisfied by anchor rings, the standard tori in E^3 , whose generating circles have radii in the ratio $1 : \sqrt{2}$. For these embeddings of the torus τ equals $2\pi^2$. White [12] has shown that τ is invariant under conformal transformations of E^3 . It has been conjectured that anchor rings whose generating circles have radii in the ratio $1 : \sqrt{2}$ and surfaces which differ from these anchor rings by conformal transformations of E^3 minimize τ among all immersions of the torus into E^3 .

There are two principal results in this paper. One states that there exists embeddings of surfaces in E^3 with arbitrary genus that satisfy the equation (2). These embeddings are the images of embedded minimal surfaces in S^3 , the standard three-sphere, under stereographic projection of S^3 onto E^3 . Lawson [10] has shown that there are minimal embeddings into S^3 of surfaces of arbi-