

Note on Dunford-Pettis Property and Schur Property

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1. Introduction. A Banach space X is said to have DP-property (*i.e.* Dunford-Pettis property) if for every weakly convergent to zero sequence (x_n) in X and for every weakly convergent to zero sequence (x_n^*) in X^* , $\lim_{n \rightarrow \infty} x_n^* x_n = 0$.

Also a Banach space X is said to possess Schur property if every weak Cauchy sequence in X is norm convergent.

Rosenthal [4] proved a fundamental result that a Banach space X contains an isomorph of ℓ_1 if and only if X admits a bounded sequence without weak Cauchy subsequences. We use this result to obtain the following:

Theorem 1. *If X has DP and is (isomorphic to) a subspace of Y^* where $Y \supseteq \ell_1$ then X has the Schur property.*

It is well known that dual WCG spaces have Radon-Nikodym property (RNP for short). By using the method of the proof analogous to that used to prove the above result we obtain a positive generalization of the result of Rosenthal [3] in the form:

Theorem 2. *Let the Banach space X satisfy DP. Then if X is isomorphic to a subspace of a conjugate Banach space that satisfies RNP then every weak Cauchy sequence in X converges in the norm topology of X .*

Our considerations along with some other people's work can be used to prove the following characterization of dual spaces with the Schur property.

Theorem 3. *Let X be a Banach space. Then X^* has the Schur property if and only if X has DP and $X \supseteq \ell_1$.*

2. Proofs.

Proof of Theorem 1. As observed by Rosenthal [3], since X is assumed to satisfy DP, given (x_n) and (f_n) sequences in X and X^* respectively such that $x_n \rightarrow 0$ weakly and (f_n) is weak Cauchy, $f_n(x_n) \rightarrow 0$.

Now we note that it is enough to show that every sequence which is weakly convergent to zero in X converges in the norm topology of X . This follows