

Composition Operators on H^p with Dense Range

RAYMOND C. ROAN

A. Introduction. Let φ be a *non-constant* analytic function mapping the unit disk \cup into itself. We will be considering the *composition operator* C_φ induced by φ and defined by

$$(C_\varphi f)(z) = (f \circ \varphi)(z) = f(\varphi(z))$$

for functions f analytic on \cup . Such composition operators are continuous on the Hardy spaces H^p , $0 < p \leq \infty$.

In [2], J. Caughran proved that if $\|\varphi\|_\infty \leq 1$ and φ maps \cup conformally onto a Carathéodory domain, then the range of C_φ is dense in H^p , $1 \leq p < \infty$. We will extend this result by proving that if $\|\varphi\|_\infty \leq 1$ and φ is a weak* generator of H^∞ , then the range of C_φ is dense in H^p , $0 < p < \infty$. We will also characterize those functions φ for which C_φ has either dense or closed range in H^∞ . Finally, we will apply our result for H^p , $0 < p < \infty$, to describe a class of domains $G \subseteq \mathbb{C}$ for which the polynomials are dense in $H^p(G)$.

B. The case p finite. The space H^∞ is a conjugate Banach space, so we can equip H^∞ with the weak* topology. For each vector subspace \mathcal{M} of H^∞ , let \mathcal{M}^1 be the subspace consisting of each point of H^∞ that is a weak* limit of a sequence of points of \mathcal{M} . Inductively, define \mathcal{M}^σ for each uncountable ordinal number σ by

$$\mathcal{M}^\sigma = [\cup \mathcal{M}^\xi]^1 \quad (\text{union over all } \xi < \sigma).$$

Since the predual of H^∞ is separable, there exists a smallest countable ordinal number σ_0 such that \mathcal{M}^{σ_0} is the weak* closure of \mathcal{M} (see [1], p. 213). The number σ_0 is called the order of \mathcal{M} .

We need the following well known fact.

Proposition 1. *A sequence $\{f_n\}$ in H^∞ converges to the function f (weak*) if and only if it is uniformly bounded and converges to f at each point of \cup .*

Definition. A function φ in H^∞ is called a *weak* generator of H^∞* if the weak* closure of the set $\mathcal{P}(\varphi) = \{P(\varphi) | P \text{ is a polynomial}\}$ is all of H^∞ . If $\mathcal{P}(\varphi)$ is of order σ , then φ is said to be a weak* generator of H^∞ of order σ .