

Weyl's Theorem for Certain Operator-Valued Potentials

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1. Introduction. The classical theorem of Weyl on the asymptotic distribution of the eigenvalues of the Laplacian has been generalized by Agmon, Browder and Gårding to elliptic operators of higher order with variable coefficients. There is a large literature on this problem, and the available formulas attain a high degree of generality and precision.

However, in the case of degenerate elliptic operators (operators whose characteristic form is merely non-negative, not strictly positive), the spectrum seems still to be little known. Indeed, in [9] it is stated that "There are many interesting unsolved problems connected with second-order equations with non-negative characteristic form, as well as with analogous equations of higher order. Among the unsolved problems we indicate the question of the spectrum of the (Dirichlet) problem." For this reason, we present here some results, proved by means of the method of random evolutions, that are quite special but that do not seem to be contained in the known results.

Let $D \subset R^n$ and $G \subset R^m$ be bounded open sets with C^∞ boundaries. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ be generic points in D and G respectively. Let Δ be the Laplacian operator on $L^2(D)$, and let A be a self-adjoint, positive-definite elliptic operator of arbitrary order $2r$ on $L^2(G)$. We write $\Delta = \Delta_x$ and $A = A_y$ to emphasize that Δ and A operate on functions of x and y respectively.

We consider operators of the form $L = -\frac{1}{2} \Delta_x - c(x)A_y$ on $L^2(D \times G)$, where $c(x)$ is non-negative. If the order of A_y is greater than 2, L is not elliptic. If A_y is of order 2, L is either elliptic or degenerate elliptic, depending on whether $c(x)$ is positive or merely non-negative. We are able to solve Cauchy's problem for the equation $u_t + Lu = 0$ and to obtain a stochastic representation for the fundamental solution of this equation as a Wiener function-space integral involving the kernel function of A_y . The asymptotic behavior of this fundamental solution for small t is easy to determine. In order for this function to be integrable asymptotically as $t \rightarrow 0$, it turns out that we need to have the following condition:

$$(1.1) \quad \int_{D \cap \{x: c(x) > 0\}} (c(x))^{-m/2r} dx < \infty.$$