

Gleason Spaces and Topological Dynamics

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1. Gleason spaces and injective envelopes. Let X be a compact Hausdorff space. Let $\Sigma = \Sigma(X)$ be the σ -algebra of subsets of X with the property of Baire and let $\eta = \eta(X)$ be the complete σ -ideal of meager subsets of X . We let Σ also denote the Banach lattice of bounded Σ -measurable functions on X with the supremum norm and we let η also denote those functions in Σ vanishing outside a meager set. It is well known [13, Theorem 15.6.15] that $f \in \Sigma$ if there is a meager set N such that f is continuous on $X \setminus N$. The Boolean quotient algebra Σ/η is known to be a complete Boolean algebra isomorphic to $\mathcal{A}(X)$, the algebra of regular open sets in X , [11]. The Stone space, X_Σ of Σ/η is isomorphic as a Banach lattice to the quotient Banach lattice Σ/η .

In [10] the space, $\mathcal{B}(X)$, of bounded functions f on X whose set, $d(f)$, of points of discontinuity belongs to η is shown to be of some importance in topological dynamics. Thus $f \in \mathcal{B}(X)$ iff the set, $c(f)$, of continuity points of f is residual. $\mathcal{B}(X)$ is given the complete seminorm $\|\cdot\|_\Sigma$ of convergence uniform on residual sets. The kernel of this semi-norm is $\mathcal{B}(X) \cap \eta$. $\mathcal{B}(X)$ is a sublattice of the Banach lattice $\mathcal{M}^\infty(X)$ of all bounded functions with the supremum norm $\|\cdot\|_X$ but $\|\cdot\|_X \geq \|\cdot\|_\Sigma$. The quotient space $\tilde{\mathcal{B}}(X) = \mathcal{B}(X)/[\mathcal{B}(X) \cap \eta]$ is an M -space of Kakutani with the quotient order and the quotient norm. It is clear that there is a Banach lattice isomorphism from $\tilde{\mathcal{B}}(X)$ into Σ/η . This isomorphism assigns to an equivalence class, \hat{f} , of $f \in \mathcal{B}(X)$ in $\tilde{\mathcal{B}}(X)$ the equivalence class of f in Σ/η . This isomorphism is actually a surjection as is shown below.

Proposition 1. $\tilde{\mathcal{B}}(X)$, $\mathcal{C}(X_\Sigma)$ and Σ/η are isomorphic as Banach lattices.

Proof. We need only find for an $f \in \Sigma$ with equivalence class $[f] \in \Sigma/\eta$, an element in $[f] \cap \mathcal{B}(X)$. Let c be a residual, hence dense, set such that f is continuous on c . For $z \in X$ let $f_1(z)$ be the limit infimum of $f(x)$ as $x \in c$ approaches z and let $f_2(z)$ be the limit supremum. f_1 is lower semicontinuous and f_2 is upper semicontinuous with $f_1 \leq f_2$ on X and $f_1 = f_2 = f$ on c . If g is any function on X satisfying $f_1 \leq g \leq f_2$ then $f_1 \leq \hat{g} \leq g \leq \check{g} \leq f_2$ where \hat{g} is the lower semi-continuous regularization of g and \check{g} is the upper semi-continuous regularization of g . Thus, $\{\hat{g} = \check{g}\} = c(g) \supset c$ so $g \in \mathcal{B}(X)$. Furthermore, $\{f = g\} \supset c$ so $f - g \in \eta$. Thus $g \in \mathcal{B}(X) \cap [f]$. One choice for g is $g = \hat{f}$. ■