

Self-adjoint Approximants

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§1. Introduction. This paper continues the research presented in [2] and [3]. A best approximation from the set of self-adjoint operators is a “self-adjoint approximant”. In [5], Halmos proved that if $B + iC$ is the representation of T with $B = B^*$ and $C = C^*$ then the distance from T to the self-adjoint operators is $\|C\|$; so B is a self-adjoint approximant. Here and throughout this paper T is a bounded linear operator on a complex Hilbert space H ; for brevity, T is said to be an “operator”. In [3], several fundamental facts about self-adjoint approximation were proved as well as a general theorem about spectral approximation of normal operators. In [2], attention centers on best approximation of subnormal and Toeplitz operators. In contrast, this paper gets detailed results about arbitrary nonnormal operators. A general reference for terms not explained is [4].

The second section of this paper proves several basic facts about the self-adjoint approximants of T , denoted $\mathcal{S}(T)$. In particular, $S \in \mathcal{S}(T)$ is related to B and C where $T = B + iC$ ($B = B^*$, $C = C^*$). The third section gives necessary and sufficient conditions for $\mathcal{S}(T)$ to be finite dimensional, as a convex set (see pp. 7–9 of [8]). The extreme points of $\mathcal{S}(T)$, are shown to be uncountable whenever the dimension of $\mathcal{S}(T)$ is more than one. Many members of $\mathcal{S}(T)$ are explicitly constructed. The final section characterizes those self-adjoint approximants S with the property that the error operator $T - S$ is normal. The extreme points of this set are characterized also.

It should be remarked that $\mathcal{S}(T)$ is a convex set which is compact in the weak operator topology. Thus, the Krein-Milman theorem shows that $\mathcal{S}(T)$ is the closed convex hull of its extreme points.

§2. Preliminaries. In this section the key properties of self-adjoint approximants are established. These results will be used repeatedly in the sequel. The starting point is an essential lemma proved in [3] (see Proposition 2 and Theorem 1).

2.1. Lemma. *Let S be a self-adjoint approximant for T , that is $S \in \mathcal{S}(T)$, where $T = B + iC$ ($B = B^*$, $C = C^*$). Then $B + (\|C\|^2 - C^2)^{1/2}$ and $B - (\|C\|^2 - C^2)^{1/2}$ belong to $\mathcal{S}(T)$, and*